CHAPTER 6 SYSTEMS OF PARTICLES AND ROTATIONAL MOTION

EXERCISES **PAGE:125**

Question 1. Give the location of the centre of mass of a (i) sphere, (ii) cylinder, (iii) ring, and (iv) cube, each of uniform mass density. Does the centre of mass of a body necessarily lie inside the body?

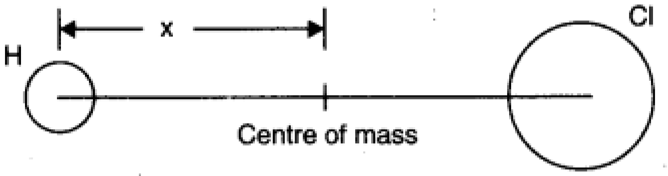
Answer: In all the four cases, as the mass density is uniform, centre of mass is located at their respective geometrical centres.

No, it is not necessary that the centre of mass of a body should lie on the body. For example, in case of a circular ring, centre of mass is at the centre of the ring, where there is no mass.

Question 2. In the HCl molecule, the separation between the nuclei of the two atoms is about 1.27 A (1 A = 10^{-10}m). Find the approximate location of the CM of the molecule, given that a chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of an atom is concentrated in its nucleus.

Answer: Let us choose the nucleus of the hydrogen atom as the origin for measuring distance. Mass of hydrogen atom,m_1= 1 unit (say) Since cholorine atom is 35.5 times as massive as hydrogen atom,

.•. mass of cholorine atom, m_2 = 35.5 units



Now, $x_1 = 0$ and $x_2 = 1.27$ Å $= 1.27 \times 10^{-10}$ m

Distance of centre of mass of HCl molecule from the origin is given by

$$X = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{1 \times 0 + 35.5 \times 1.27 \times 10^{-10}}{1 + 35.5} \text{ m}$$

$$= \frac{35.5 \times 1.27}{36.5} \times 10^{-10} \text{m} = 1.235 \times 10^{-10}\text{m} = 1.235 \text{ Å}$$

Question 3. A child sits stationary at one end of a long trolley moving uniformly with a speed V on a smooth horizontal floor. If the child gets up and runs about on the trolley in any manner, what is the speed of the CM of the (trolley + child) system?

Answer: When the child gets up and runs about on the trolley, the speed of the centre of mass of the trolley and child remains unchanged irrespective of the manner of motion of child. It is because here child and trolley constitute one single system and forces involved are purely internal forces. As there is no external force, there is no change in momentum of the system and velocity remains unchanged.

Question 4.

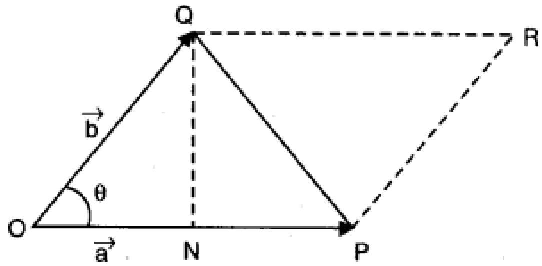
Show that the area of the triangle contained between the vectors $\vec{a}$ and $\vec{b}$ is one half of the magnitude of $\vec{a} \times \vec{b}$.

Answer:

Let $\vec{a}$ be represented $\overrightarrow{OP}$ and $\vec{b}$ be represented by $\overrightarrow{OQ}$. Let $\angle POQ = \theta$, Fig.

Complete the $\parallel$ gm $OPRQ$. Join PQ.

Draw $QN \perp OP$



In ΔOQN, $\sin\theta = \frac{QN}{OQ} = \frac{QN}{b}$

$QN = b \sin\theta$

Now, by definition, $|\vec{a} \times \vec{b}| = ab \sin\theta = (OP)(QN)$

$= \frac{2(OP)(QN)}{2} = 2 \times$ area of ΔOPQ

$\therefore$ area of $\Delta OPQ = \frac{1}{2}|\vec{a} \times \vec{b}|$, which was to be proved.

Question 5.

Show that $\vec{a}.(\vec{b} \times \vec{c})$ is equal in magnitude to the volume of the parallelopiped formed on the three vectors, $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Answer:

Let a parallelopiped be formed on the three vectors.

$\overrightarrow{OA} = \vec{a}, \quad \overrightarrow{OB} = \vec{b}$

and $\overrightarrow{OC} = \vec{c}$

Now, $\vec{b} \times \vec{c} = bc \sin 90° \ \hat{n} = bc\ \hat{n}$

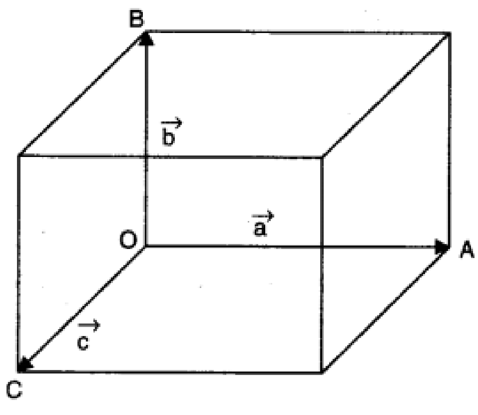
where $\hat{n}$ is unit vector along $\overrightarrow{OA}$ perpendicular to the plane containing $\vec{b}$ and $\vec{c}$.

Now $\vec{a}.(\vec{b} \times \vec{c}) = \vec{a}.\, bc\ \hat{n}$

$= (a)(bc) \cos 0°$

$= a\ b\ c$

which is equal in magnitude to the volume of the parallelopiped.



Question 6. Find the components along the x, y, z-axes of the angular momentum l of a particle, whose position vector is r with components x, y, z and momentum is p with components p_x, p_y and p_z. Show that if the particle moves only in the x-y plane

the angular momentum has only a z- component.

Answer:

We know that angular momentum $\vec{l}$ of a particle having position vector $\vec{r}$ and momentum $\vec{p}$ is given by

$$\vec{l} = \vec{r} \times \vec{p}$$

But $\vec{r} = [x\hat{i} + y\hat{j} + z\hat{k}]$, where x, y, z are the components of

$$\vec{r} \text{ and } \vec{p} = [p_x\hat{i} + p_y\hat{j} + p_z\hat{k}]$$

$$\therefore \quad \vec{l} = \vec{r} \times \vec{p} = [x\hat{i} + y\hat{j} + z\hat{k}] \times [p_x\hat{i} + p_y\hat{j} + p_z\hat{k}]$$

or

$$(l_x\hat{i} + l_y\hat{j} + l_z\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$= (yp_z - zp_y)\hat{i} + (zp_x - xp_z)\hat{j} + (xp_y - yp_x)\hat{k}$$

From this relation, we conclude that

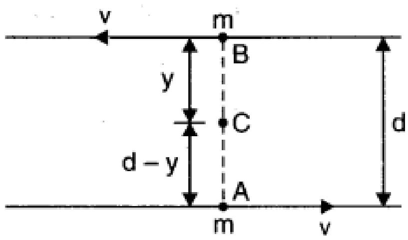
$$l_x = yp_z - zp_y, \quad l_y = zp_x - xp_z \quad \text{and} \quad l_z = xp_y - yp_x$$

If the given particle moves only in the $x - y$ plane, then $z = 0$ and $p_z = 0$ and hence, $\vec{l} = (xp_y - yp_x)\hat{k}$, which is only the z-comonent of $\vec{l}$.

It means that for a particle moving only in the $x - y$ plane, the angular momentum has only the z-component.

Question 7. Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the vector angular momentum of the two particle system the same whatever be the point about which the angular momentum is taken.

Answer:



Angular momentum about A,

$$L_A = mv \times 0 + mv \times d$$

$$= mvd$$

Angular momentum about B,

$$L_B = mv \times d + mv \times 0$$

$$= mvd$$

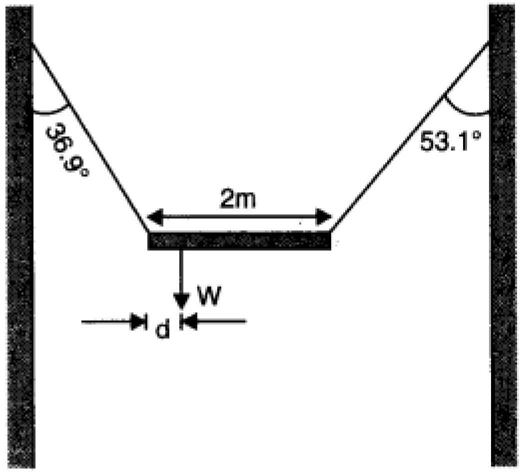
Angular momentum about C,

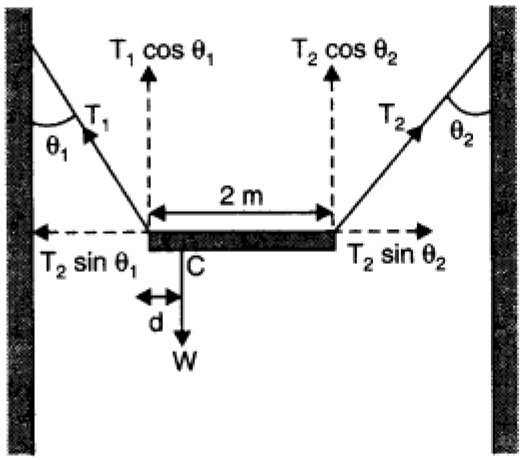
$$L_c = mv \times y + mv \times (d - y) = mvd$$

In all the three cases, the direction of angular momentum is the same.

$$\therefore \quad \vec{L}_A = \vec{L}_B = \vec{L}_C.$$

Question 8. A non-uniform bar of weight W is suspended at rest by two strings of negligible weight as shown in Fig. The angles made by the strings with the vertical are 36.9° and ' 53.2° respectively. The bar is 2 m long. Calculate the distance d of the centre of gravity of the bar from its left end.





Answer:

As it is clear from Fig.,

$$\theta_1 = 36.9°, \quad \theta_2 = 53.1°.$$

If T_1, T_2 are the tensions in the two strings, then for equilibrium along the horizontal,

$$T_1 \sin\theta_1 = T_2 \sin\theta_2$$

or $$\frac{T_1}{T_2} = \frac{\sin\theta_2}{\sin\theta_1} = \frac{\sin 53.1°}{\sin 36.9°}$$

$$= \frac{0.7404}{0.5477} = 1.3523$$

Let d be the distance of centre of gravity C of the bar from the left end.

For rotational equilibrium about C,

$$T_1 \cos\theta_1 \times d = T_2 \cos\theta_2 (2 - d)$$

$$T_1 \cos 36.9° \times d = T_2 \cos 53.1° (2 - d)$$

$$T_1 \times 0.8366\, d = T_2 \times 0.6718 (2 - d)$$

Put $T_1 = 13523\ T_2$ and solve to get

$$d = 0.745 \text{ m}$$

$$\tau = I_1\alpha_1 = I_2\alpha_2$$

or $$\frac{\alpha_1}{\alpha_2} = \frac{I_2}{I_1} = \frac{(2/5)MR^2}{MR^2} = \frac{2}{5}$$

or $$\alpha_2 = \frac{5}{2}\alpha_1.$$

Question 9. A car weighs 1800 kg. The distance between its front and back axles is 1.8 m. Its centre of gravity is 1.05 m behind the front axle. Determine the force exerted by the level ground on each front wheel and each back wheel.

Answer: Let F_1 and F_2 be the forces exerted by the level ground on front wheels and back wheels respectively.Considering rotational equilibrium about the front wheels,F_2 x 1.8 = mg x 1.05 or F_2 = 1.05/1.8 x 1800 x 9.8 N =10290 N Force on each back wheel is =10290/2 N or 5145 N.

Considering rotational equilibrium about the back wheels.

F_1 x 1.8 = mg (1.8 – 1.05) = 0.75 x 1800 x 9.8

or F_1=0.75 x 1800 x 9.8/1.8 = 7350 N

Force on each front wheel is 7350/2 N or 3675 N.

Question 10 Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry, and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time?

Answer: Let M be the mass and R the radius of the hollow cylinder, and also of the solid sphere. Their moments of inertia about the respective axes are $I_1 = MR^2$ and $I_2 = 2/5\ MR^2$

Let τ be the magnitude of the torque applied to the cylinder and the sphere, producing angular accelerations $\alpha_1$and α_2 respectively. Then $\tau = I_1\ \alpha_1 = I_2\ \alpha_2$

The angular acceleration 04 produced in the sphere is larger. Hence, the sphere will acquire larger angular speed after a given time.

Question 11 A solid cylinder of mass 20 kg rotates about its axis with angular speed 100 rad s^{-1}. The radius of the cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of angular momentum of the cylinder about its axis?

Answer: M = 20 kg

Angular speed, w = 100 rad s^{-1}; R = 0.25 m

Moment of inertia of the cylinder about its axis

$=1/2\ MR^2 = 1/2 \times 20\ (0.25)^2\ kg\ m^2 = 0.625\ kg\ m^2$

Rotational kinetic energy,

$E_r = 1/2\ Iw2 = 1/2 \times 0.625 \times (100)^2\ J = 3125\ J$

Angular momentum,

$L = I_w = 0.625 \times 100\ J_s = 62.5\ J_s$.

Question 12(a) A child stands at the centre of a turntable with his arms outstretched. The turntable is set rotating with an angular speed of 40 rev/min. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to 2/5 times the initial value? Assume that the turntable rotates without friction,

(b) Show that the child's new kinetic energy of rotation is more than the initial kinetic energy of rotation. How do you account of this increase in kinetic energy?

Answer: (a) Suppose, initial moment of inertia of the child is I1 Then final moment of inertia,

$$I_2 = \frac{2}{5} I_1$$

Also, $\nu_1 = 40 \text{ rev min}^{-1}$

By using the principle of conservation of angular momentum, we get

$$I_1\omega_1 = I_2\omega_2 \quad \text{or} \quad I_1(2\pi\nu_1) = I_2(2\pi\nu_2)$$

or

$$\nu_2 = \frac{I_1\nu_1}{I_2} = \frac{I_1 \times 40}{\frac{2}{5} \times I_1} = 100 \text{ rev min}^{-1}$$

(*b*)

$$\frac{\text{Final K.E. of rotation}}{\text{Initial K.E. of rotation}} = \frac{\frac{1}{2}I_2\omega_2^2}{\frac{1}{2}I_1\omega_1^2} = \frac{\frac{1}{2}I_2(2\pi\nu_2)^2}{\frac{1}{2}I_1(2\pi\nu_1)^2} = \frac{I_2\nu_2^2}{I_1\nu_1^2} = \frac{\frac{2}{5}I_1 \times (100)^2}{\frac{2}{5}I_1 \times (40)^2} = 2.5$$

Clearly, final $(K.E.)_{rot}$ becomes more because the child uses his internal energy when he folds his hands to increase the kinetic energy.

Question 13 A rope of negligible mass is wound round a hollow cylinder of mass 3 kg and radius 40 cm. What is the angular acceleration of the cylinder if the rope is pulled with a force of 30 N? What is the linear acceleration of the rope? Assume that there is no slipping.

Answer: Here, M = 3 kg, R = 40 cm = 0.4 m

Moment of inertia of the hollow cylinder about its axis.

$I = MR^2 = 3(0.4)^2 = 0.48 \text{ kg m}^2$

Force applied $F = 30$ N

∴ Torque, $\tau = F \times R = 30 \times 0.4 = 12$ N–m.

If α is angular acceleration produced, then from $\tau = I\alpha$

$$\alpha = \frac{\tau}{I} = \frac{12}{0.48} = 25 \text{ rad s}^{-2}$$

Linear acceleration, $a = R\alpha = 0.4 \times 25 = 10 \text{ ms}^{-2}$.

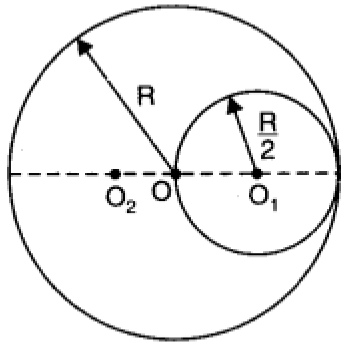
Question 14 To maintain a rotor at a uniform angular speed of 200 rad s^{-1}, an engine needs to transmit a torque of 180 Nm. What is the power required by the engine? Note: (Uniform angular velocity in the absence of friction implies zero torque. In practice, applied torque is needed to counter frictional torque). Assume that the engine is 100 efficient.

Answer: Here, a = 200 rad s^{-1}; Torque, τ= 180 N-m

Since,Power, P = Torque (τ) x angular speed (w)

= 180 x 200 = 36000 watt = 36 KW.

Question 15 From a uniform disk of radius R, a circular hole of radius R/2 is cut out. The centre of the hole is at R/2 from the centre of the original disc.Locate the centre of gravity of the resulting flat body.



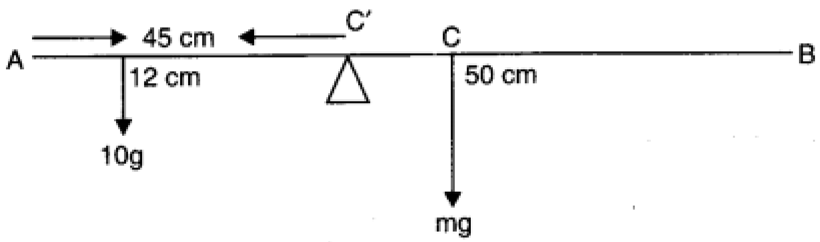
Answer: Let from a bigger uniform disc of radius R with centre O a smaller circular hole of radius R/2 with its centre at O_1 (where R OO_1 = R/2) is cut out. Let centre of gravity or the centre of mass of remaining flat body be at O_2, where OO_2 = x. If σ be mass per unit area, then mass of whole disc $M_1 = \pi R^2 \sigma$ and mass of cut out part

$$M_2 = \pi\left(\frac{R}{2}\right)^2 \sigma = \frac{1}{4}\pi R^2 \sigma = \frac{M_1}{4}$$

$$\therefore \quad x = \frac{M_1 \times (0) - M_2(OO_1)}{M_1 - M_2} = \frac{0 - \frac{M_1}{4} \times \frac{R}{2}}{M_1 - \frac{M_1}{4}} = -\frac{R}{6}$$

i.e., O_2 is at a distance R/6 from centre of disc on diametrically opposite side to centre of hole.

Question 16 A metre stick is balanced on a knife edge at its centre. When two coins, each of mass 5g are put one on top of the other at the 12.0 cm mark, the stick is found to be balanced at 45.0 cm. What is the mass of the metre stick?
Answer: Let m be the mass of the stick concentrated at C, the 50 cm mark, see fig.



For equilibrium about C, the 45 cm mark,
10 g (45 – 12) = mg (50 – 45)
10 g x 33 = mg x 5

=> m = 10 x 33/5
or m = 66 grams.

Question 17 The oxygen molecule has a mass of 5.30 x 10^{-26} kg and a moment of inertia of 1.94 x 10^{-45} kg m^2 about an axis through its centre perpendicular to the lines joining the two atoms. Suppose the mean speed of such a molecule in a gas is 500 m/s and that its kinetic energy of rotation is two thirds of its kinetic energy of translation. Find the average angular velocity of the molecule.

Answer:

Here, $m = 5.30 \times 10^{-26}$ kg

$I = 1.94 \times 10^{-46}$ kg m^2

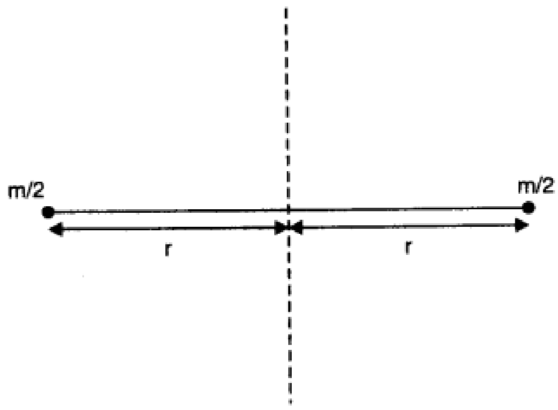
$v = 500$ m/s

If $\frac{m}{2}$ is mass of each atom of oxygen and $2r$ is distance between the two atoms as shown in Fig, then

$$I = \frac{m}{2}r^2 + \frac{m}{2}r^2 = mr^2$$

$$r = \sqrt{\frac{I}{m}} = \sqrt{\frac{1.94 \times 10^{-46}}{5.30 \times 10^{-26}}}$$

$$= 0.61 \times 10^{-10} \text{ m}$$



As K.E. of rotation $= \frac{2}{3}$ K.E. of translation

$$\therefore \quad \frac{1}{2}I\omega^2 = \frac{2}{3} \times \frac{1}{2}m\omega^2$$

$$\frac{1}{2}(mr^2)\omega^2 = \frac{1}{2}mv^2$$

$$\omega = \sqrt{\frac{2}{3}}\frac{v}{r} = \sqrt{\frac{2}{3}} \times \frac{500}{0.61 \times 10^{-10}} = 6.7 \times 10^{12} \text{ rad/s.}$$