Exercise 4.1 Page: 81

Evaluate the following determinants in Exercise 1 and 2.

Question 1. $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

Solution

$\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$ = 2(-1) – 4(-5) = -2 + 20 = 18

Question 2. (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

(ii) $\begin{vmatrix} x^2 - x - 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Solution

(i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

= (cosθ)(cosθ) – (-sinθ) (sinθ)= $\cos^2\theta + \sin^2\theta$= 1

(ii) $\begin{vmatrix} x^2 - x - 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

= $(x^2 - x + 1)(x + 1) - (x - 1)(x + 1)$

= $x^3 - x^2 + x + x^2 - x + 1 - (x^2 - 1)$

= $x^3 + 1 - x^2 + 1$

= $x^3 - x^2 + 2$

Question 3. If A = $\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$ **then show that |2A| = 4|A|**

Solution

Given: A = $\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$

then 2A = **2 x** $\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$

$$\therefore 2A = 2\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$$

$$\therefore \text{L.H.S.} = |2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = 2\times4-4\times8 = 8-32 = -24$$

$$\text{Now, } |A| = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 1\times2-2\times4 = 2-8 = -6$$

$$\therefore \text{R.H.S.} = 4|A| = 4\times(-6) = -24$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

Hence, proved.

Question 4. If A = $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ **then show that 3|A| = 27|A|**

Solution

Given: A = $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ then 3A =**3** $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

It can be observed that in the first column, two entries are zero. Thus, we expand along the first column (C_1) for easier calculation.

$$|A| = 1\begin{vmatrix}1 & 2\\0 & 4\end{vmatrix} - 0\begin{vmatrix}0 & 1\\0 & 4\end{vmatrix} + 0\begin{vmatrix}0 & 1\\1 & 2\end{vmatrix} = 1(4-0) - 0 + 0 = 4$$

$$\therefore 27|A| = 27(4) = 108 \qquad \text{...(i)}$$

$$\text{Now, } 3A = 3\begin{bmatrix}1 & 0 & 1\\0 & 1 & 2\\0 & 0 & 4\end{bmatrix} = \begin{bmatrix}3 & 0 & 3\\0 & 3 & 6\\0 & 0 & 12\end{bmatrix}$$

$$\therefore |3A| = 3\begin{vmatrix}3 & 6\\0 & 12\end{vmatrix} - 0\begin{vmatrix}0 & 3\\0 & 12\end{vmatrix} + 0\begin{vmatrix}0 & 3\\3 & 6\end{vmatrix}$$

$$= 3(36-0) = 3(36) = 108 \qquad \text{...(ii)}$$

From equations (i) and (ii), we have:

$$|3A| = 27|A|$$

Hence, proved.

Question 5. Evaluate the determinants:

(i) $\begin{vmatrix}3 & -1 & -2\\0 & 0 & -1\\3 & -5 & 0\end{vmatrix}$

(ii) $\begin{vmatrix}0 & 1 & 2\\-1 & 0 & -3\\-2 & 3 & 0\end{vmatrix}$

(iii) $\begin{vmatrix}3 & -4 & 5\\1 & 1 & -2\\2 & 3 & 1\end{vmatrix}$

(iv) $\begin{vmatrix}2 & -1 & -2\\0 & 2 & -1\\3 & -5 & 0\end{vmatrix}$

Solution

Evaluate the determinants:

(i) Given: $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

It can be observed that in the second row, two entries are zero. Thus, we expand along the second row for easier calculation.

= $|A| = -0\begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} + 0\begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix} = (-15+3) = -12$

(ii) Given: $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$

By expanding along the first row, we have:

=
$$\begin{aligned} |A| &= 3\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \\ &= 3(1+6) + 4(1+4) + 5(3-2) \\ &= 3(7) + 4(5) + 5(1) \\ &= 21 + 20 + 5 = 46 \end{aligned}$$

(iii) Given: $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

Expanding along first row,

=
$$\begin{aligned} |A| &= 0\begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & -3 \\ -2 & 0 \end{vmatrix} + 2\begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix} \\ &= 0 - 1(0-6) + 2(-3-0) \\ &= -1(-6) + 2(-3) \end{aligned}$$

= 0 + 6 – 6 = 0

(iv) Given: $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Expanding along first row,

= $|A| = 2\begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} - 0\begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} + 3\begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix}$
$= 2(0-5) - 0 + 3(1+4)$

= -10 + 15 = 5

Question 6. If A = $\begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$ find |A|

Solution

Given: A = $\begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$

Expanding along first row,

= $|A| = 1\begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix} - 1\begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}$
$= 1(-9+12) - 1(-18+15) - 2(8-5)$
$= 1(3) - 1(-3) - 2(3)$
$= 3 + 3 - 6$
$= 6 - 6$
$= 0$

Question7. Find the value of x if:

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

(ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

Solution

(i) Given: $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

⇒ 2 x 1 – 5 x 4 = 2x * x – 6 x 4

⇒ 2 – 20 = $2x^2$ – 24

⇒ $2x^2$ = 6

⇒ x^2 = 3

⇒ x = ± √3

(ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

⇒ 2 x 5 – 4 x 3 = x * 5 – 2x – 3

⇒10 – 12 = 5x – 6x

⇒ – 2 = -x

⇒ x = 2

Question 8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$ **then x is equal to:**

(A) 6

(B) ± 6

(C) – 6

(D) 0

Solution

Given: $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$

⇒x * x – 18 x 2 = 6 x 6 – 18 x 2

⇒x^2 – 36 = 36 – 36

⇒x^2 – 36 = 0

⇒x = ± 6

Therefore, option (B) is correct.

Exercise 4.2 Page: 83

Question 1. Find the area of the triangle with vertices at the points given in each of the following:

(i) (1, 0), (6, 0), (4, 3)

(ii) (2, 7), (1, 1), (10, 8)

(iii) (−2, −3), (3, 2), (−1, −8)

Solution :

(i) The area of the triangle with vertices (1, 0), (6, 0), (4, 3) is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[1(0-3) - 0(6-4) + 1(18-0)\right]$$

$$= \frac{1}{2}[-3+18] = \frac{15}{2} \text{ square units}$$

(ii) The area of the triangle with vertices (2, 7), (1, 1), (10, 8) is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[2(1-8)-7(1-10)+1(8-10)\right]$$

$$= \frac{1}{2}\left[2(-7)-7(-9)+1(-2)\right]$$

$$= \frac{1}{2}[-14+63-2] = \frac{1}{2}[-16+63]$$

= $$= \frac{47}{2} \text{ square units}$$

(iii) The area of the triangle with vertices (−2, −3), (3, 2), (−1, −8)

$$\Delta = \frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[-2(2+8)+3(3+1)+1(-24+2)\right]$$

$$= \frac{1}{2}\left[-2(10)+3(4)+1(-22)\right]$$

$$= \frac{1}{2}[-20+12-22]$$

= $$= -\frac{30}{2} = -15$$

Question 2. Show that the points A(a,b + c), B(b, c + a), C(c, a+b) are collinear.

Solution :

Area of ΔABC is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b-a & a-b & 0 \\ c-a & a-c & 0 \end{vmatrix} \quad (\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1)$$

$$= \frac{1}{2}(a-b)(c-a)\begin{vmatrix} a & b+c & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= \frac{1}{2}(a-b)(c-a)\begin{vmatrix} a & b+c & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad (\text{Applying } R_3 \to R_3 + R_2)$$

$$= 0 \qquad (\text{All elements of } R_3 \text{ are } 0)$$

Therefore, points A, B and C are collinear.

Question 3. Find values of k if area of triangle is 4 sq. units and vertices are:

(i) (*k*, 0), (4, 0), (0, 2)

(ii) (−2, 0), (0, 4), (0, *k*)

Solution :

We know that the area of a triangle whose vertices are (x_1, y_1), (x_2, y_2), and

(x_3, y_3) is the absolute value of the determinant (Δ), where

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

It is given that the area of triangle is 4 square units.

$\therefore \Delta = \pm 4$.

(i) The area of the triangle with vertices (*k*, 0), (4, 0), (0, 2) is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[k(0-2) - 0(4-0) + 1(8-0)\right]$$
$$= \frac{1}{2}[-2k+8] = -k+4$$

$\therefore -k + 4 = \pm 4$

When $-k + 4 = -4$, $k = 8$.

When $-k + 4 = 4$, $k = 0$.

Hence, $k = 0, 8$.

(ii) The area of the triangle with vertices (−2, 0), (0, 4), (0, *k*) is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[-2(4-k)\right]$$
$$= k-4$$

$\therefore k - 4 = \pm 4$

When $-k + 4 = -4$, $k = 8$.

When $-k + 4 = 4$, $k = 0$.

Hence, $k = 0, 8$.

(ii) The area of the triangle with vertices (−2, 0), (0, 4), (0, *k*) is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[-2(4-k)\right]$$
$$= k - 4$$

$\therefore k - 4 = \pm 4$

When $k - 4 = -4$, $k = 0$.

When $k - 4 = 4$, $k = 8$.

Hence, $k = 0, 8$.

Question4. (i) Find the equation of the line joining (1, 2) and (3, 6) using determinants.

(ii) Find the equation of the line joining (3, 1) and (9, 3) using determinants.

Solution
(i) Let P(x, y) be any point on the line joining the points (1, 2) and (3, 6).

Then, Area of triangle that could be formed by these points is zero.

$$\therefore \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix} = 0$$
$$\Rightarrow \frac{1}{2}\left[1(6-y) - 2(3-x) + 1(3y-6x)\right] = 0$$
$$\Rightarrow 6 - y - 6 + 2x + 3y - 6x = 0$$
$$\Rightarrow 2y - 4x = 0$$
$$\Rightarrow y = 2x$$

Hence, the equation of the line joining the given points is $y = 2x$.

(ii) Let P (x, y) be any point on the line joining points A (3, 1) and

B (9, 3). Then, the points A, B, and P are collinear. Therefore, the area of triangle ABP will be zero.

$$\therefore \frac{1}{2}\begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$
$$\Rightarrow \frac{1}{2}\left[3(3-y)-1(9-x)+1(9y-3x)\right]=0$$
$$\Rightarrow 9-3y-9+x+9y-3x=0$$
$$\Rightarrow 6y-2x=0$$
$$\Rightarrow x-3y=0$$

Hence, the equation of the line joining the given points is $x - 3y = 0$.

Question 5. If area of triangle is 35 square units with vertices (2, −6), (5, 4), and (*k*, 4). Then *k* is

(A). 12

(B). −2

(C). −12, −2

(D). 12, −2

Solution :

The area of the triangle with vertices (2, −6), (5, 4), and (*k*, 4) is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[2(4-4)+6(5-k)+1(20-4k)\right]$$

$$= \frac{1}{2}\left[30-6k+20-4k\right]$$

$$= \frac{1}{2}\left[50-10k\right]$$

$$= 25-5k$$

It is given that the area of the triangle is ±35.

Therefore, we have:

⇒ 25 – 5k = ± 35

⇒ 5(5 – k) = ± 35

⇒ 5 – k = ± 7

When 5 − *k* = −7, *k* = 5 + 7 = 12.

When 5 − *k* = 7, *k* = 5 − 7 = −2.

Hence, *k* = 12, −2.

The correct answer is D.

Therefore, option (D) is correct.

Exercise 4.3 Page: 87

Question 1. Write minors and cofactors of the elements of the following determinants:

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

(ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Solution

(i) Let $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

Minor of element a_{ij} is M_{ij}.

$\therefore M_{11}$ = minor of element $a_{11} = 3$

M_{12} = minor of element $a_{12} = 0$

M_{21} = minor of element $a_{21} = -4$

M_{22} = minor of element $a_{22} = 2$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.

$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (0) = 0$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-4) = 4$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (2) = 2$

(ii) Let $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Minor of element a_{ij} is M_{ij}.

$\therefore M_{11}$ = minor of element $a_{11} = d$

M_{12} = minor of element $a_{12} = b$

M_{21} = minor of element $a_{21} = c$

M_{22} = minor of element $a_{22} = a$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.

$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (d) = d$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (b) = -b$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (c) = -c$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (a) = a$

Question 2. Write minors and cofactors of the elements of the following determinants:

(i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Solution

(i) The given determinant is $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$.

By the definition of minors and cofactors, we have:

M_{11} = minor of a_{11}= $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

M_{12} = minor of a_{12}= $\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

M_{13} = minor of a_{13} = $\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$

M_{21} = minor of a_{21} = $\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

M_{22} = minor of a_{22} = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

M_{23} = minor of a_{23} = $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

M_{31} = minor of a_{31}= $\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$

M_{32} = minor of a_{32} = $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

M_{33} = minor of a_{33} = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

A_{11} = cofactor of a_{11}= $(-1)^{1+1}$ M_{11} = 1

A_{12} = cofactor of a_{12} = $(-1)^{1+2}$ M_{12} = 0

A_{13} = cofactor of a_{13} = $(-1)^{1+3}$ M_{13} = 0

A_{21} = cofactor of a_{21} = $(-1)^{2+1}$ M_{21} = 0

A_{22} = cofactor of a_{22} = $(-1)^{2+2}$ M_{22} = 1

A_{23} = cofactor of a_{23} = $(-1)^{2+3}$ M_{23} = 0

A_{31} = cofactor of a_{31} = $(-1)^{3+1}$ M_{31} = 0

A_{32} = cofactor of a_{32} = $(-1)^{3+2}$ M_{32} = 0

A_{33} = cofactor of a_{33} = $(-1)^{3+3}$ M_{33} = 1

(ii) The given determinant is $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$.

By definition of minors and cofactors, we have:

M_{11} = minor of a_{11} = $\begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10 + 1 = 11$

M_{12} = minor of a_{12} = $\begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6$

M_{13} = minor of a_{13} = $\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$

M_{21} = minor of a_{21} = $\begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 4 = -4$

M_{22} = minor of a_{22} = $\begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$

M_{23} = minor of a_{23} = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

M_{31} = minor of a_{31} = $\begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0 - 20 = -20$

M_{32} = minor of a_{32} = $\begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13$

M_{33} = minor of a_{33} = $\begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5$

A_{11} = cofactor of a_{11} = $(-1)^{1+1}$ M_{11} = 11

A_{12} = cofactor of $a_{12} = (-1)^{1+2} M_{12} = -6$

A_{13} = cofactor of $a_{13} = (-1)^{1+3} M_{13} = 3$

A_{21} = cofactor of $a_{21} = (-1)^{2+1} M_{21} = 4$

A_{22} = cofactor of $a_{22} = (-1)^{2+2} M_{22} = 2$

A_{23} = cofactor of $a_{23} = (-1)^{2+3} M_{23} = -1$

A_{31} = cofactor of $a_{31} = (-1)^{3+1} M_{31} = -20$

A_{32} = cofactor of $a_{32} = (-1)^{3+2} M_{32} = 13$

A_{33} = cofactor of $a_{33} = (-1)^{3+3} M_{33} = 5$

Question 3. Using cofactors of elements of second row, evaluate: $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

Solution :

The given determinant is $\begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

We have:

$M_{21} = \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = 9 - 16 = -7$

$\therefore A_{21}$ = cofactor of $a_{21} = (-1)^{2+1} M_{21} = 7$

$M_{22} = \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = 15 - 8 = 7$

$\therefore A_{22}$ = cofactor of $a_{22} = (-1)^{2+2} M_{22} = 7$

$M_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7$

$\therefore A_{23}$ = cofactor of $a_{23} = (-1)^{2+3} M_{23} = -7$

We know that Δ is equal to the sum of the product of the elements of the second row with their corresponding cofactors.

$\therefore \Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = 2(7) + 0(7) + 1(-7) = 14 - 7 = 7$

Question 4. Using cofactors of elements of third column, evaluate: $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

Solution

The given determinant is $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

We have:

$M_{13} = \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = z - y$

$M_{23} = \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = z - x$

$M_{33} = \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = y - x$

$\therefore A_{13}$ = cofactor of $a_{13} = (-1)^{1+3} M_{13} = (z - y)$

A_{23} = cofactor of $a_{23} = (-1)^{2+3} M_{23} = -(z - x) = (x - z)$

A_{33} = cofactor of $a_{33} = (-1)^{3+3} M_{33} = (y - x)$

We know that Δ is equal to the sum of the product of the elements of the second row with their corresponding cofactors.

$$\begin{aligned}
\therefore \Delta &= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} \\
&= yz(z-y) + zx(x-z) + xy(y-x) \\
&= yz^2 - y^2z + x^2z - xz^2 + xy^2 - x^2y \\
&= (x^2z - y^2z) + (yz^2 - xz^2) + (xy^2 - x^2y) \\
&= z(x^2 - y^2) + z^2(y-x) + xy(y-x) \\
&= z(x-y)(x+y) + z^2(y-x) + xy(y-x) \\
&= (x-y)[zx + zy - z^2 - xy] \\
&= (x-y)[z(x-z) + y(z-x)] \\
&= (x-y)(z-x)[-z+y] \\
&= (x-y)(y-z)(z-x)
\end{aligned}$$

Hence, $\Delta = (x-y)(y-z)(z-x)$.

Question 5. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is Cofactors of a_{ij}, then value of Δ is given by

(A) $a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33}$

(B) $a_{11} A_{11} + a_{12} A_{21} + a_{13} A_{31}$

(C) $a_{21} A_{11} + a_{22} A_{12} + a_{23} A_{13}$

(D) $a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31}$

Solution :

We know that:

Δ = Sum of the product of the elements of a column (or a row) with their corresponding cofactors

$\therefore \Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Hence, the value of Δ is given by the expression given in alternative **D**. Option (D) is correct.

Exercise 4.4 Page: 92

Find adjoint of each of the matrices in Exercise 1 and 2.

Question1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

We have,

$A_{11} = 4,\ A_{12} = -3,\ A_{21} = -2,\ A_{22} = 1$

$\therefore adjA = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Question2. $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$.

Solution

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$.

We have,

$A_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$

$A_{12} = -\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -(2 + 10) = -12$

$A_{13} = \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 0 + 6 = 6$

$A_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1 - 0) = 1$

$A_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1 + 4 = 5$

$A_{23} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -(0 - 2) = 2$

$A_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5 - 6 = -11$

$A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5 - 4) = -1$

$A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 + 2 = 5$

Hence, $adjA = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$.

Question 3.

Verify *A* (*adj A*) = (*adj A*) *A* = |A| *I* .

$$\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

Solution

$$A=\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}$$

we have,

$$|A|=-12-(-12)=-12+12=0$$

$$\therefore |A|I=0\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Now,

$$A_{11}=-6, A_{12}=4, A_{21}=-3, A_{22}=2$$

$$\therefore adjA=\begin{bmatrix}-6 & -3\\ 4 & 2\end{bmatrix}$$

Now,

$$A(adjA)=\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}\begin{bmatrix}-6 & -3\\ 4 & 2\end{bmatrix}$$

$$=\begin{bmatrix}-12+12 & -6+6\\ 24-24 & 12-12\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Also, $(adjA)A=\begin{bmatrix}-6 & -3\\ 4 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}$

$$=\begin{bmatrix}-12+12 & -18+18\\ 8-8 & 12-12\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Hence, $A(adjA)=(adjA)A=|A|I.$

Question 4.

Verify *A* (*adj A*) = (*adj A*) *A* = |A| *I* .

$$\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & -2\\ 1 & 0 & 3\end{bmatrix}$$

Solution

Let A = $\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & -2\\ 1 & 0 & 3\end{bmatrix}$

$$|A| = 1(0-0)+1(9+2)+2(0-0) = 11$$

$$\therefore |A|I = 11\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}11 & 0 & 0\\0 & 11 & 0\\0 & 0 & 11\end{bmatrix}$$

Now,

$A_{11} = 0, A_{12} = -(9+2) = -11, A_{13} = 0$

$A_{21} = -(-3-0) = 3, A_{22} = 3-2 = 1, A_{23} = -(0+1) = -1$

$A_{31} = 2-0 = 2, A_{32} = -(-2-6) = 8, A_{33} = 0+3 = 3$

$$\therefore adjA = \begin{bmatrix}0 & 3 & 2\\-11 & 1 & 8\\0 & -1 & 3\end{bmatrix}$$

Now,

$$A(adjA) = \begin{bmatrix}1 & -1 & 2\\3 & 0 & -2\\1 & 0 & 3\end{bmatrix}\begin{bmatrix}0 & 3 & 2\\-11 & 1 & 8\\0 & -1 & 3\end{bmatrix}$$

$$= \begin{bmatrix}0+11+0 & 3-1-2 & 2-8+6\\0+0+0 & 9+0+2 & 6+0-6\\0+0+0 & 3+0-3 & 2+0+9\end{bmatrix}$$

$$= \begin{bmatrix}11 & 0 & 0\\0 & 11 & 0\\0 & 0 & 11\end{bmatrix}$$

Also,

$$(adjA)\cdot A = \begin{bmatrix}0 & 3 & 2\\-11 & 1 & 8\\0 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & -1 & 2\\3 & 0 & -2\\1 & 0 & 3\end{bmatrix}$$

$$= \begin{bmatrix}0+9+2 & 0+0+0 & 0-6+6\\-11+3+8 & 11+0+0 & -22-2+24\\0-3+3 & 0+0+0 & 0+2+9\end{bmatrix}$$

$$= \begin{bmatrix}11 & 0 & 0\\0 & 11 & 0\\0 & 0 & 11\end{bmatrix}$$

Hence, $A(adjA) = (adjA)A = |A|I.$

Find the inverse of the matrix (if it exists) given in Exercise 5 to 11.

Question 5. $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

Solution :

Let $A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$.

we have,

$|A| = 6 + 8 = 14$

Now,

$A_{11} = 3, A_{12} = -4, A_{21} = 2, A_{22} = 2$

$\therefore adjA = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|} adjA = \frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

Question6. $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

Solution :

Let $A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$.

we have,

$|A| = -2 + 15 = 13$

Now,

$A_{11} = 2, A_{12} = 3, A_{21} = -5, A_{22} = -1$

$\therefore adjA = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|} adjA = \frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

Question 7. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Solution :

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$.

We have,

$|A| = 1(10-0) - 2(0-0) + 3(0-0) = 10$

Now,

$A_{11} = 10 - 0 = 10, A_{12} = -(0-0) = 0, A_{13} = 0 - 0 = 0$

$A_{21} = -(10-0) = -10, A_{22} = 5 - 0 = 5, A_{23} = -(0-0) = 0$

$A_{31} = 8 - 6 = 2, A_{32} = -(4-0) = -4, A_{33} = 2 - 0 = 2$

$\therefore adjA = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$

$\therefore \ A^{-1} = \frac{1}{|A|} adjA = \frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$

Question 8. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

Solution :

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$.

We have,

$|A| = 1(-3-0) - 0 + 0 = -3$

Now,

$A_{11} = -3-0 = -3, A_{12} = -(-3-0) = 3, A_{13} = 6-15 = -9$

$A_{21} = -(0-0) = 0, A_{22} = -1-0 = -1, A_{23} = -(2-0) = -2$

$A_{31} = 0-0 = 0, A_{32} = -(0-0) = 0, A_{33} = 3-0 = 3$

$$\therefore adjA = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

$$\therefore \ A^{-1} = \frac{1}{|A|} adjA = -\frac{1}{3}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

Question 9. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

Solution :

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$.

We have,

$$|A| = 2(-1-0) - 1(4-0) + 3(8-7)$$
$$= 2(-1) - 1(4) + 3(1)$$
$$= -2 - 4 + 3$$
$$= -3$$

Now,

$A_{11} = -1 - 0 = -1, A_{12} = -(4-0) = -4, A_{13} = 8 - 7 = 1$

$A_{21} = -(1-6) = 5, A_{22} = 2 + 21 = 23, A_{23} = -(4+7) = -11$

$A_{31} = 0 + 3 = 3, A_{32} = -(0-12) = 12, A_{33} = -2 - 4 = -6$

$$\therefore adjA = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = -\frac{1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

Question 10. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

Solution

Let A = $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

By expanding along C_1, we have:

$$|A| = 1(8-6) - 0 + 3(3-4) = 2 - 3 = -1$$

Now,

$$A_{11} = 8-6 = 2, A_{12} = -(0+9) = -9, A_{13} = 0-6 = -6$$
$$A_{21} = -(-4+4) = 0, A_{22} = 4-6 = -2, A_{23} = -(-2+3) = -1$$
$$A_{31} = 3-4 = -1, A_{32} = -(-3-0) = 3, A_{33} = 2-0 = 2$$

$$\therefore adjA = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$\therefore \ A^{-1} = \frac{1}{|A|} adjA = -\begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

Question 11. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

Solution

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}.$$

We have,

$$|A| = 1(-\cos^2\alpha - \sin^2\alpha) = -(\cos^2\alpha + \sin^2\alpha) = -1$$

Now,

$$A_{11} = -\cos^2\alpha - \sin^2\alpha = -1, A_{12} = 0, A_{13} = 0$$
$$A_{21} = 0, A_{22} = -\cos\alpha, A_{23} = -\sin\alpha$$
$$A_{31} = 0, A_{32} = -\sin\alpha, A_{33} = \cos\alpha$$

$$\therefore adjA = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$\therefore \ A^{-1} = \frac{1}{|A|} \cdot adjA = -\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$$

Question 12. Let $A=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}$ and $B=\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$. Verify that $(AB)^{-1}=B^{-1}A^{-1}$

Solution

Let $A=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}$.

We have,

$|A|=15-14=1$

Now,

$A_{11}=5, A_{12}=-2, A_{21}=-7, A_{22}=3$

$\therefore adjA=\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$

$\therefore A^{-1}=\frac{1}{|A|}\cdot adjA=\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$

Now, let $B=\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$.

We have,

$|B|=54-56=-2$

$\therefore adjB=\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}$

$\therefore B^{-1}=\frac{1}{|B|}adjB=-\frac{1}{2}\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}=\begin{bmatrix}-\frac{9}{2} & 4\\ \frac{7}{2} & -3\end{bmatrix}$

Now,

$$B^{-1}A^{-1}=\begin{bmatrix}-\frac{9}{2} & 4\\ \frac{7}{2} & -3\end{bmatrix}\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{45}{2}-8 & \frac{63}{2}+12\\ \frac{35}{2}+6 & -\frac{49}{2}-9\end{bmatrix}=\begin{bmatrix}-\frac{61}{2} & \frac{87}{2}\\ \frac{47}{2} & -\frac{67}{2}\end{bmatrix} \qquad ...(1)$$

Then,

$$AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix}$$

$$= \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$$

Therefore, we have $|AB| = 67 \times 61 - 87 \times 47 = 4087 - 4089 = -2.$

Also,

$$adj(AB) = \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

$$\therefore (AB)^{-1} = \frac{1}{|AB|} adj(AB) = -\frac{1}{2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{61}{2} & \frac{87}{2} \\ \frac{47}{2} & -\frac{67}{2} \end{bmatrix} \quad ...(2)$$

From (1) and (2), we have:

$(AB)^{-1} = B^{-1}A^{-1}$

Hence, the given result is proved.

Question 13. If A = $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that A2 – 5A + 7I = 0. Hence find A^{-1}

Solution

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$\therefore A^2 - 5A + 7I$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, $A^2 - 5A + 7I = O.$

$$\therefore A \cdot A - 5A = -7I$$

$$\Rightarrow A \cdot A\left(A^{-1}\right) - 5AA^{-1} = -7IA^{-1} \qquad \left[\text{Post-multiplying by } A^{-1} \text{ as } |A| \neq 0\right]$$

$$\Rightarrow A\left(AA^{-1}\right) - 5I = -7A^{-1}$$

$$\Rightarrow AI - 5I = -7A^{-1}$$

$$\Rightarrow A^{-1} = -\frac{1}{7}(A - 5I)$$

$$\Rightarrow A^{-1} = \frac{1}{7}(5I - A)$$

$$= \frac{1}{7}\left(\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\right) = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

Question 14. For the matrix A = $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ **find numbers a and b such that** $A^2 + aA + bI = O$**.**

Solution

$$A=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$$

$$\therefore A^2=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}=\begin{bmatrix}9+2 & 6+2\\ 3+1 & 2+1\end{bmatrix}=\begin{bmatrix}11 & 8\\ 4 & 3\end{bmatrix}$$

Now,

$$A^2+aA+bI=O$$

$$\Rightarrow (AA)A^{-1}+aAA^{-1}+bIA^{-1}=O \qquad \left[\text{Post-multiplying by } A^{-1} \text{ as } |A|\neq 0\right]$$

$$\Rightarrow A\left(AA^{-1}\right)+aI+b\left(IA^{-1}\right)=O$$

$$\Rightarrow AI+aI+bA^{-1}=O$$

$$\Rightarrow A+aI=-bA^{-1}$$

$$\Rightarrow A^{-1}=-\frac{1}{b}(A+aI)$$

Now,

$$A^{-1}=\frac{1}{|A|}adjA=\frac{1}{1}\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}=\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}$$

We have:

$$\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}=-\frac{1}{b}\left(\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}+\begin{bmatrix}a & 0\\ 0 & a\end{bmatrix}\right)=-\frac{1}{b}\begin{bmatrix}3+a & 2\\ 1 & 1+a\end{bmatrix}=\begin{bmatrix}\frac{-3-a}{b} & -\frac{2}{b}\\ -\frac{1}{b} & \frac{-1-a}{b}\end{bmatrix}$$

Comparing the corresponding elements of the two matrices, we have:

$$-\frac{1}{b}=-1\Rightarrow b=1$$

$$\frac{-3-a}{b}=1\Rightarrow -3-a=1\Rightarrow a=-4$$

Hence, −4 and 1 are the required values of *a* and *b* respectively.

Question 15. For the matrix A = $\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$**, show that** $A^3-6A^2+5A+11I=O$**. Hence, find** A^{-1}**.**

Solution

$$A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$$

$$A^2=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1+1+2 & 1+2-1 & 1-3+3\\ 1+2-6 & 1+4+3 & 1-6-9\\ 2-1+6 & 2-2-3 & 2+3+9\end{bmatrix}=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}$$

$$A^3=A^2\cdot A=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}4+2+2 & 4+4-1 & 4-6+3\\ -3+8-28 & -3+16+14 & -3-24-42\\ 7-3+28 & 7-6-14 & 7+9+42\end{bmatrix}$$

$$=\begin{bmatrix}8 & 7 & 1\\ -23 & 27 & -69\\ 32 & -13 & 58\end{bmatrix}$$

$\therefore A^3-6A^2+5A+11I$

$$=\begin{bmatrix}8 & 7 & 1\\ -23 & 27 & -69\\ 32 & -13 & 58\end{bmatrix}-6\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}+5\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}+11\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}8 & 7 & 1\\ -23 & 27 & -69\\ 32 & -13 & 58\end{bmatrix}-\begin{bmatrix}24 & 12 & 6\\ -18 & 48 & -84\\ 42 & -18 & 84\end{bmatrix}+\begin{bmatrix}5 & 5 & 5\\ 5 & 10 & -15\\ 10 & -5 & 15\end{bmatrix}+\begin{bmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{bmatrix}$$

$$=\begin{bmatrix}24 & 12 & 6\\ -18 & 48 & -84\\ 42 & -18 & 84\end{bmatrix}-\begin{bmatrix}24 & 12 & 6\\ -18 & 48 & -84\\ 42 & -18 & 84\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=O$$

Thus, $A^3-6A^2+5A+11I=O$.

Now,

$A^3-6A^2+5A+11I=O$

$\Rightarrow (AAA)A^{-1}-6(AA)A^{-1}+5AA^{-1}+11IA^{-1}=0$ [Post-multiplying by A^{-1} as $|A|\neq 0$]

$\Rightarrow AA(AA^{-1})-6A(AA^{-1})+5(AA^{-1})=-11(IA^{-1})$

$\Rightarrow A^2-6A+5I=-11A^{-1}$

$\Rightarrow A^{-1}=-\frac{1}{11}(A^2-6A+5I)$ ...(1)

Now,

$A^2-6A+5I$

$$=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}-6\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}+5\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}-\begin{bmatrix}6 & 6 & 6\\ 6 & 12 & -18\\ 12 & -6 & 18\end{bmatrix}+\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}$$

$$=\begin{bmatrix}9 & 2 & 1\\ -3 & 13 & -14\\ 7 & -3 & 19\end{bmatrix}-\begin{bmatrix}6 & 6 & 6\\ 6 & 12 & -18\\ 12 & -6 & 18\end{bmatrix}$$

$$=\begin{bmatrix}3 & -4 & -5\\ -9 & 1 & 4\\ -5 & 3 & 1\end{bmatrix}$$

From equation (1), we have:

$$A^{-1}=-\frac{1}{11}\begin{bmatrix}3 & -4 & -5\\ -9 & 1 & 4\\ -5 & 3 & 1\end{bmatrix}=\frac{1}{11}\begin{bmatrix}-3 & 4 & 5\\ 9 & -1 & -4\\ 5 & -3 & -1\end{bmatrix}$$

Question 16. If A = $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1}

Solution

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now,

$A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 40 & -30 & 30 \\ -30 & 40 & -30 \\ 30 & -30 & 40 \end{bmatrix} - \begin{bmatrix} 40 & -30 & 30 \\ -30 & 40 & -30 \\ 30 & -30 & 40 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 9A - 4I = O$

Now,

$A^3 - 6A^2 + 9A - 4I = O$

$\Rightarrow (AAA)A^{-1} - 6(AA)A^{-1} + 9AA^{-1} - 4IA^{-1} = O$ [Post-multiplying by A^{-1} as $|A| \neq 0$]

$\Rightarrow AA(AA^{-1}) - 6A(AA^{-1}) + 9(AA^{-1}) = 4(IA^{-1})$

$\Rightarrow AAI - 6AI + 9I = 4A^{-1}$

$\Rightarrow A^2 - 6A + 9I = 4A^{-1}$

$\Rightarrow A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$ ...(1)

$A^2 - 6A + 9I$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

From equation (1), we have:

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Question 17. Let A be a non-singular matrix of order 3 x 3. Then |adjA| is equal to:

(A) |A|

(B) |A|^2

(C) |A|^3

(D) 3|A|

Solution

$$(adjA)A = |A|I = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$\Rightarrow |(adjA)A| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$

$$\Rightarrow |adjA||A| = |A|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = |A|^3 (I)$$

$$\therefore |adjA| = |A|^2$$

Therefore, option (B) is correct.

Question 18. If A is an invertible matrix of order 2, then det (A^{-1}) is equal to:

(A) det A

(B) 1/det A

(C) 1

(D) 0

Solution :

Since A is an invertible matrix, A^{-1} exists and $A^{-1} = \frac{1}{|A|} adj A.$

As matrix A is of order 2, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then, $|A| = ad - bc$ and $adj A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Now,

$$A^{-1} = \frac{1}{|A|} adj A = \begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix}$$

$$\therefore |A^{-1}| = \begin{vmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{vmatrix} = \frac{1}{|A|^2} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix} = \frac{1}{|A|^2}(ad - bc) = \frac{1}{|A|^2} . |A| = \frac{1}{|A|}$$

$$\therefore \det(A^{-1}) = \frac{1}{\det(A)}$$

Therefore, option (B) is correct.

Exercise 4.5 Page: 97

Examine the consistency of the system of equations in Exercises 1 to 3.

Question 1.

***x* + 2*y* = 2**

2*x* + 3*y* = 3

Solution
Matrix form of given equations is AX = B

$$A=\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix},\ X=\begin{bmatrix}x\\ y\end{bmatrix} \text{ and } B=\begin{bmatrix}2\\ 3\end{bmatrix}.$$

Now,

$$|A| = 1(3) - 2(2) = 3 - 4 = -1 \neq 0$$

∴ *A* is non-singular.

Therefore, A^{-1} exists.

Question 2.

2*x* − *y* = 5

***x* + *y* = 4**

Solution
Matrix form of given equations is AX = B

$$A=\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix},\ X=\begin{bmatrix}x\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}5\\ 4\end{bmatrix}.$$

Now,

$$|A| = 2(1) - (-1)(1) = 2 + 1 = 3 \neq 0$$

∴ *A* is non-singular.

Therefore, A^{-1} exists.

Hence, the given system of equations is consistent.

Hence, the given system of equations is consistent.

Question 3.

***x* + 3*y* = 5**

2*x* + 6*y* = 8

Solution
Matrix form of given equations is AX = B

$$A=\begin{bmatrix}1 & 3\\ 2 & 6\end{bmatrix},\ X=\begin{bmatrix}x\\ y\end{bmatrix} \text{ and } B=\begin{bmatrix}5\\ 8\end{bmatrix}.$$

Now,

$$|A|=1(6)-3(2)=6-6=0$$

∴ *A* is a singular matrix.

$$(adjA)=\begin{bmatrix}6 & -3\\ -2 & 1\end{bmatrix}$$

$$(adjA)B=\begin{bmatrix}6 & -3\\ -2 & 1\end{bmatrix}\begin{bmatrix}5\\ 8\end{bmatrix}=\begin{bmatrix}30-24\\ -10+8\end{bmatrix}=\begin{bmatrix}6\\ -2\end{bmatrix}\neq O$$

Thus, the solution of the given system of equations does not exist. Hence, the system of equations is inconsistent.

Examine the consistency of the system of equations in Exercises 4 to 6.

Question 4.

***x* + *y* + *z* = 1**

2*x* + 3*y* + 2*z* = 2

***ax* + *ay* + 2*az* = 4**

Solution
Matrix form of given equations is AX = B

$$A=\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 2\\ a & a & 2a\end{bmatrix},\ X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}1\\ 2\\ 4\end{bmatrix}$$

Now,

$$|A|=1(6a-2a)-1(4a-2a)+1(2a-3a)$$
$$=4a-2a-a=4a-3a=a\neq 0$$

∴ *A* is non-singular.

Therefore, A^{-1} exists.

Hence, the given system of equations is consistent.

Question 5.

$3x - y - 2z = 2$

$2y - z = -1$

$3x - 5y = 3$

Solution
Matrix form of given equations is AX = B

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Now,

$$|A| = 3(0-5) - 0 + 3(1+4) = -15 + 15 = 0$$

$\therefore$ A is a singular matrix.

Now,

$$(adjA) = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$$

$$\therefore (adjA)B = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -10-10+15 \\ -6-6+9 \\ -12-12+18 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \neq O$$

Thus, the solution of the given system of equations does not exist. Hence, the system of equations is inconsistent.

Question6.

$5x - y + 4z = 5$

$2x + 3y + 5z = 2$

$5x - 2y + 6z = -1$

Solution
Matrix form of given equations is AX = B

$$A=\begin{bmatrix}5 & -1 & 4\\ 2 & 3 & 5\\ 5 & -2 & 6\end{bmatrix},\ X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}5\\ 2\\ -1\end{bmatrix}.$$

Now,

$$\begin{aligned}|A| &= 5(18+10)+1(12-25)+4(-4-15)\\ &= 5(28)+1(-13)+4(-19)\\ &= 140-13-76\\ &= 51 \neq 0\end{aligned}$$

∴ A is non-singular.

Therefore, A^{-1} exists.

Hence, the given system of equations is consistent.

Solve the system of linear equations, using matrix method, in Exercise 7 to 10.

Question7.

5x + 2y =4

7x + 3y = 5

Solution
Matrix form of given equations is AX = B

$A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

Now, $|A| = 15 - 14 = 1 \neq 0$.

Thus, A is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1} = \frac{1}{|A|}(adjA)$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12-10 \\ -28+25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Hence, $x = 2$ and $y = -3$.

Question8.

2x – y = – 2

3x + 4y = 3

Solution

Matrix form of given equations is AX = B

$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

Now,

$|A| = 8 + 3 = 11 \neq 0$

Thus, A is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -5 \\ 12 \end{bmatrix} = \begin{bmatrix} -\frac{5}{11} \\ \frac{12}{11} \end{bmatrix}$$

Hence, $x = \frac{-5}{11}$ and $y = \frac{12}{11}$.

Question9.

4x – 3y = 3

3x – 5y = 7

Solution

Matrix form of given equations is AX = B

$$A=\begin{bmatrix}4 & -3\\ 3 & -5\end{bmatrix},\ X=\begin{bmatrix}x\\ y\end{bmatrix} \text{ and } B=\begin{bmatrix}3\\ 7\end{bmatrix}.$$

Now,

$$|A|=-20+9=-11\neq 0$$

Thus, *A* is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1}=\frac{1}{|A|}(adjA)=-\frac{1}{11}\begin{bmatrix}-5 & 3\\ -3 & 4\end{bmatrix}=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}$$

$$\therefore X=A^{-1}B=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}\begin{bmatrix}3\\ 7\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}\begin{bmatrix}3\\ 7\end{bmatrix}=\frac{1}{11}\begin{bmatrix}15-21\\ 9-28\end{bmatrix}=\frac{1}{11}\begin{bmatrix}-6\\ -19\end{bmatrix}=\begin{bmatrix}-\frac{6}{11}\\ -\frac{19}{11}\end{bmatrix}$$

Hence, $x=\frac{-6}{11}$ and $y=\frac{-19}{11}$.

Question10.

5*x* + 2*y* = 3

3*x* + 2*y* = 5

Solution
Matrix form of given equations is AX = B

$$A=\begin{bmatrix}5 & 2\\ 3 & 2\end{bmatrix},X=\begin{bmatrix}x\\ y\end{bmatrix} \text{ and } B=\begin{bmatrix}3\\ 5\end{bmatrix}$$

Now,

$$|A|=10-6=4\neq 0$$

Thus, *A* is non-singular. Therefore, its inverse exists.

Solve the system of linear equations, using matrix method, in Exercise 11 to 14.

Question11.

$2x + y + z = 1$

$x - 2y - z = \frac{3}{2}$

$3y - 5z = 9$

Solution

Matrix form of given equations is AX = B

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}.$$

Now,

$$|A| = 2(10+3) - 1(-5-3) + 0 = 2(13) - 1(-8) = 26 + 8 = 34 \neq 0$$

Thus, *A* is non-singular. Therefore, its inverse exists.

Now, $A_{11} = 13, A_{12} = 5, A_{13} = 3$

$A_{21} = 8, A_{22} = -10, A_{23} = -6$

$A_{31} = 1, A_{32} = 3, A_{33} = -5$

$$\therefore\ A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

$$\therefore\ X = A^{-1}B = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13+12+9 \\ 5-15+27 \\ 3-9-45 \end{bmatrix}$$

$$= \frac{1}{34}\begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$$

Hence, $x = 1, y = \frac{1}{2}$, and $z = -\frac{3}{2}$.

Question12.

***x* − *y* + *z* = 4**

2*x* + *y* − 3*z* = 0

***x* + *y* + *z* = 2**

Solution

Matrix form of given equations is AX = B

$$A=\begin{bmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{bmatrix},\ X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}4\\ 0\\ 2\end{bmatrix}.$$

Now,

$$|A|=1(1+3)+1(2+3)+1(2-1)=4+5+1=10\neq 0$$

Thus, *A* is non-singular. Therefore, its inverse exists.

Now, $A_{11}=4, A_{12}=-5, A_{13}=1$

$A_{21}=2, A_{22}=0, A_{23}=-2$

$A_{31}=2, A_{32}=5, A_{33}=3$

$$\therefore\ A^{-1}=\frac{1}{|A|}(adjA)=\frac{1}{10}\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{bmatrix}$$

$$\therefore X=A^{-1}B=\frac{1}{10}\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{bmatrix}\begin{bmatrix}4\\ 0\\ 2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{10}\begin{bmatrix}16+0+4\\ -20+0+10\\ 4+0+6\end{bmatrix}$$

$$=\frac{1}{10}\begin{bmatrix}20\\ -10\\ 10\end{bmatrix}$$

$$=\begin{bmatrix}2\\ -1\\ 1\end{bmatrix}$$

Hence, $x=2, y=-1,$ and $z=1$.

Question13.

2x + 3y + 3z = 5

x − 2y + z = −4

3x − y − 2z = 3

Solution

Matrix form of given equations is AX = B

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}.$$

Now,

$$|A| = 2(4+1) - 3(-2-3) + 3(-1+6) = 2(5) - 3(-5) + 3(5) = 10 + 15 + 15 = 40 \neq 0$$

Thus, A is non-singular. Therefore, its inverse exists.

Now, $A_{11} = 5, A_{12} = 5, A_{13} = 5$

$A_{21} = 3, A_{22} = -13, A_{23} = 11$

$A_{31} = 9, A_{32} = 1, A_{33} = -7$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence, $x = 1, y = 2,$ and $z = -1$.

Question14.

***x* − *y* + 2*z* = 7**

3*x* + 4*y* − 5*z* = −5

2*x* − *y* + 3*z* = 12

Solution

Matrix form of given equations is AX = B

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}.$$

Now,

$$|A| = 1(12-5) + 1(9+10) + 2(-3-8) = 7 + 19 - 22 = 4 \neq 0$$

Thus, *A* is non-singular. Therefore, its inverse exists.

Now, $A_{11} = 7, A_{12} = -19, A_{13} = -11$

$A_{21} = 1, A_{22} = -1, A_{23} = -1$

$A_{31} = -3, A_{32} = 11, A_{33} = 7$

$$\therefore\ A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Hence, $x = 2, y = 1,$ and $z = 3$.

Question15. If A = $\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$ **find A^{-1}. Using A^{-1} solve the system of equations**

$$\begin{aligned} 2x - 3y + 5z &= 11 \\ 3x + 2y - 4z &= -5 \\ x + y - 2z &= -3 \end{aligned}$$

Solution

$$A=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}$$

$$\therefore |A| = 2(-4+4)+3(-6+4)+5(3-2) = 0-6+5 = -1 \neq 0$$

Now, $A_{11}=0,\ A_{12}=2,\ A_{13}=1$

$A_{21}=-1,\ A_{22}=-9,\ A_{23}=-5$

$A_{31}=2,\ A_{32}=23,\ A_{33}=13$

$$\therefore\ A^{-1}=\frac{1}{|A|}(adjA)=-\begin{bmatrix}0 & -1 & 2\\ 2 & -9 & 23\\ 1 & -5 & 13\end{bmatrix}=\begin{bmatrix}0 & 1 & -2\\ -2 & 9 & -23\\ -1 & 5 & -13\end{bmatrix} \quad ...(1)$$

Now, the given system of equations can be written in the form of *AX* = *B*, where

$$A=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}11\\ -5\\ -3\end{bmatrix}.$$

The solution of the system of equations is given by $X = A^{-1}B$.

$$X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0 & 1 & -2\\ -2 & 9 & -23\\ -1 & 5 & -13\end{bmatrix}\begin{bmatrix}11\\ -5\\ -3\end{bmatrix} \qquad [\text{Using } (1)]$$

$$=\begin{bmatrix}0-5+6\\ -22-45+69\\ -11-25+39\end{bmatrix}$$

$$=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$

Hence, $x=1,\ y=2,$ and $z=3$.

Question16. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ` 60. The cost of 2 kg onion, 4 kg wheat and 2 kg rice is ` 90. The cost of 6 kg onion, 2 k wheat and 3 kg rice is ` 70. Find cost of each item per kg by matrix method.

Solution :

Let the cost of onions, wheat, and rice per kg be Rs *x*, Rs *y*,and Rs *z* respectively.

Then, the given situation can be represented by a system of equations as:

$4x + 3y + 2z = 60$

$2x + 4y + 6z = 90$

$6x + 2y + 3z = 70$

This system of equations can be written in the form of $AX = B$, where

$$A=\begin{bmatrix}4 & 3 & 2\\ 2 & 4 & 6\\ 6 & 2 & 3\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}60\\ 90\\ 70\end{bmatrix}.$$

$$|A|=4(12-12)-3(6-36)+2(4-24)=0+90-40=50\neq 0$$

Now, $A_{11}=0, A_{12}=30, A_{13}=-20$

$A_{21}=-5, A_{22}=0, A_{23}=10$

$A_{31}=10, A_{32}=-20, A_{33}=10$

$$\therefore adjA=\begin{bmatrix}0 & -5 & 10\\ 30 & 0 & -20\\ -20 & 10 & 10\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}adjA=\frac{1}{50}\begin{bmatrix}0 & -5 & 10\\ 30 & 0 & -20\\ -20 & 10 & 10\end{bmatrix}$$

Now,

$X = A^{-1} B$

$$\Rightarrow X=\frac{1}{50}\begin{bmatrix}0 & -5 & 10\\ 30 & 0 & -20\\ -20 & 10 & 10\end{bmatrix}\begin{bmatrix}60\\ 90\\ 70\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{50}\begin{bmatrix}0-450+700\\ 1800+0-1400\\ -1200+900+700\end{bmatrix}$$

$$=\frac{1}{50}\begin{bmatrix}250\\ 400\\ 400\end{bmatrix}$$

$$=\begin{bmatrix}5\\ 8\\ 8\end{bmatrix}$$

$\therefore x=5, y=8, \text{and } z=8.$

Hence, the cost of onions is Rs 5 per kg, the cost of wheat is Rs 8 per kg, and the cost of rice is Rs 8 per kg.

Chapter 4 Miscellaneous

1. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

Ans. Let $\Delta = \begin{bmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{bmatrix}$

Expanding along first row,

$$\Delta = x\begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin\theta\begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x \end{vmatrix} + \cos\theta\begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1 \end{vmatrix}$$

$\Rightarrow$ $\Delta = x(-x^2-1) - \sin\theta(-x\sin\theta - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta)$

$\Rightarrow$ $\Delta = -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$

$\Rightarrow$ $\Delta = -x^3 - x + x(\sin^2\theta + \cos^2\theta)$ = $-x^3 - x + x$ = $-x^3$ which is independent of θ.

2. Without expanding the determinants, prove that: $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$.

Ans. L.H.S. = $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

Multiplying R_1 by a, R_2 by b and R_3 by c, $\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$ = $\dfrac{abc}{abc}\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$

= $\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$ = $-\begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix}$ [Interchanging C_1 and C_3]

= $(-)(-)\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$ = $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$ [Interchanging C_2 and C_3]
Proved.

2. Evaluate: $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Ans. Let $\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Expanding along first row,

= $\cos\alpha\cos\beta(\cos\alpha\cos\beta-0)-\cos\alpha\sin\beta(-\cos\alpha\sin\beta-0)-\sin\alpha\left(-\sin\alpha\sin^2\beta-\sin\alpha\cos^2\beta\right)$

= $\cos^2\alpha\cos^2\beta+\cos^2\alpha\sin^2\beta+\sin^2\alpha\left(\sin^2\beta+\cos^2\beta\right)$

= $\cos^2\alpha\left(\cos^2\beta+\sin^2\beta\right)+\sin^2\alpha\left(\sin^2\beta+\cos^2\beta\right)$

= $\cos^2\alpha+\sin^2\alpha$

= 1

3. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$.

Ans. Given: $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Since, $(AB)^{-1} = B^{-1}A^{-1}$ [Reversal law] ....(i)

Now $|B| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$

= $1(3-0)-2(-1-0)+(-2)(2-0)$ = $3+2-4=1 \neq 0$

Therefore, B^{-1} exists.

$\therefore$ $B_{11}=3, B_{12}=1, B_{13}=2$ and $B_{21}=2, B_{22}=1, B_{23}=2$ and $B_{31}=6, B_{32}=2, B_{33}=5$

$\therefore$ adj. B = $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}'$ = $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

$\therefore$ $B^{-1} = \frac{1}{|B|}(\text{adj. } B) = \frac{1}{1}\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

From eq. (i), $(AB)^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

$\Rightarrow$ $(AB)^{-1} = \begin{bmatrix} 9-30+30 & -3+12-12 & 3-10+12 \\ 3-15+10 & -1+6-4 & 1-5+4 \\ 6-30+25 & -2+12-10 & 2-10+10 \end{bmatrix}$

$= \begin{bmatrix} 9 & -3 & 5 \\ -2 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

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4. Let A = $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, verify that:

(i) $(\text{adj. A})^{-1} = \text{adj. } (A^{-1})$

(ii) $(A^{-1})^{-1} = A$

Ans. Given: Matrix A = $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$\therefore |A| = \begin{vmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{vmatrix}$

$\Rightarrow |A| = 1(15-1)-(-2)(-10-1)+1(-2-3) = 14-22-5 = -13 \neq 0$

Therefore, A^{-1} exists.

$\therefore A_{11} = 14, A_{12} = 11, A_{13} = -5$ and $A_{21} = 11, A_{22} = 4, A_{23} = -3$

and $A_{31} = -5, A_{32} = -3, A_{33} = -1$

$\therefore$ adj. A = $\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$ = B (say)

$\therefore A^{-1} = \frac{1}{|A|}(\text{adj. A}) = \frac{-1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$ ....(i)

$\Rightarrow$ $|B| = \begin{vmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{vmatrix}$

= $14(-4-9)-11(-11-15)-5(-33+20)$ = $169 \neq 0$

Therefore, B^{-1} exists.

$\therefore$ $B_{11} = -13, B_{12} = 26, B_{13} = -13$ and $B_{21} = 26, B_{22} = -39, B_{23} = -13$

and $B_{31} = -13, B_{32} = -13, B_{33} = -65$

$\therefore$ adj. B = $\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$ = $-13\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$\Rightarrow$ $B^{-1} = (\text{adj. A})^{-1} = \frac{1}{|B|}(\text{adj. B})$

= $\frac{1}{169}(-13)\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

= $\frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ ....(ii)

Now to find $\text{adj. } A^{-1} = \text{adj. } C$ (say), where

C = $A^{-1} = \frac{-1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$

$$= \begin{bmatrix} -14/13 & -11/13 & 5/13 \\ -11/13 & -4/11 & 3/13 \\ 5/13 & 3/11 & 1/13 \end{bmatrix}$$

C = $A^{-1} = \frac{-14}{13}\left(\frac{-4}{169}-\frac{9}{169}\right)-\left(\frac{-11}{13}\right)\left(\frac{-11}{169}-\frac{15}{169}\right)+\frac{5}{13}\left(\frac{-33}{169}+\frac{20}{169}\right)$

C = $A^{-1} = \frac{-14}{13}\left(\frac{-13}{169}\right)+\frac{11}{13}\left(\frac{-26}{169}\right)+\frac{5}{13}\left(\frac{-13}{169}\right)$ $= \frac{14}{169}-\frac{22}{169}-\frac{5}{169}$ $= \frac{-13}{169}=\frac{-1}{13}\neq 0$

Therefore, C^{-1} exists.

$\therefore$ $C_{11}=\frac{-1}{13}, C_{12}=\frac{2}{13}, C_{13}=\frac{-1}{13}$ and $C_{21}=\frac{2}{13}, C_{22}=\frac{-3}{13}, C_{23}=\frac{-1}{13}$

and $C_{31}=\frac{-1}{13}, C_{32}=\frac{-1}{13}, C_{33}=\frac{-5}{13}$

$\therefore$ adj. A = $\text{adj.}\left(A^{-1}\right)=\begin{vmatrix} -1/13 & 2/13 & -1/13 \\ 2/13 & -3/13 & -1/13 \\ -1/13 & -1/13 & -5/13 \end{vmatrix}$

$= \frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ ....(iii)

Again $\left(A^{-1}\right)^{-1} = C^{-1} = \frac{1}{|C|}(\text{adj. } C)$

$= \frac{1}{-1/13}\left(\frac{-1}{13}\right)\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ = A (given)

(i) $(\text{adj. A})^{-1} = \text{adj.}\left(A^{-1}\right)$

$\Rightarrow \frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

[From eq. (ii) and (iii)]

(ii) $\left(A^{-1}\right)^{-1} = A$

$\Rightarrow \frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

5. Evaluate: $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

Ans. Let $\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

$= \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix} \left[R_1 \to R_1 + R_2 + R_3\right]$

$= 2(x+y)\begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

$= 2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x+y-y & x-y \\ x+y & x-x-y & y-x-y \end{vmatrix} \left[C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1\right]$

$= 2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x & x-y \\ x+y & -y & -x \end{vmatrix}$

$= 2(x+y).1\begin{vmatrix} x & x-y \\ -y & -x \end{vmatrix}$

$= 2(x+y)\{-x^2+y(x-y)\}$

$= 2(x+y)(-x^2+xy-y^2)$

$= -2(x+y)(x^2-xy+y^2)$

$= -2(x^3+y^3)$

6. Evaluate: $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Ans. Let $\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

$= \begin{vmatrix} 1 & x & y \\ 0 & x+y-x & 0 \\ 0 & 0 & x+y-y \end{vmatrix} [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

$= \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix} = 1\begin{vmatrix} y & 0 \\ 0 & x \end{vmatrix} = xy$

7. Solve the system of the following equations: (Using matrices):

$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4; \quad \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1; \quad \frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$

Ans. Putting $\frac{1}{x} = u, \frac{1}{y} = v$ and $\frac{1}{z} = w$ in the given equations,
$2u + 3v + 10w = 4$; $4u - 6v + 5w = 1$; $6u + 9v - 20w = 2$

$\therefore$ the matrix form of given equations is $\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ [AX= B]

Here, A = $\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$, X = $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ and B = $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$\therefore |A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$

= $2(120-45) - 3(-80-30) + 10(36+36)$

= $150 + 330 + 750 = 1200 \neq 0$

$\therefore$ A^{-1} exists and unique solution is $X = A^{-1}B$ ....(i)

Now $A_{11} = 75, A_{12} = 110, A_{13} = 72$ and $A_{21} = 150, A_{22} = -100, A_{23} = 0$

and $A_{31} = 75, A_{32} = 30, A_{33} = -24$

$\therefore$ adj. A = $\begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}'$ = $\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

And $A^{-1} = \frac{\text{adj.A}}{|A|} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

$\therefore$ From eq. (i),

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 1/3 \\ 1/5 \end{bmatrix}$$

$$\therefore u = \frac{1}{2}, v = \frac{1}{3}, w = \frac{1}{5}$$

$$\Rightarrow x = \frac{1}{u} = 2, y = \frac{1}{v} = 3, z = \frac{1}{w} = 5$$

8. If x, y, z are non-zero real numbers, then the inverse of matrix A $= \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is:

(A) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{1} \end{bmatrix}$

(B) $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{1} \end{bmatrix}$

(C) $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

(D) $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ans. Given: Matrix A = $\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

$\therefore$ $|A| = \begin{vmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{vmatrix}$

$\Rightarrow$ $|A| = x(yz - 0) - 0 + 0 = xyz \neq 0$

$\therefore$ A^{-1} exists and unique solution is $X = A^{-1}B$ ....(i)

Now $A_{11} = yz, A_{12} = 0, A_{13} = 0$ and $A_{21} = 0, A_{22} = xz, A_{23} = 0$ and $A_{31} = 0, A_{32} = 0, A_{33} = xy$

$\therefore$ adj. A = $\begin{vmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{vmatrix}$ = $\begin{vmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{vmatrix}$

And $A^{-1} = \frac{\text{adj.}A}{|A|} = \frac{1}{xyz}\begin{vmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{vmatrix}$

= $\begin{vmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xz}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{vmatrix}$

$= \begin{vmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{vmatrix}$

$= \begin{vmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{vmatrix}$

Therefore, option (A) is correct.

9. Let A = $\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\pi$. Then:

(A) Det (A) = 0

(B) Det (A) $\in (2, \infty)$

(C) Det (A) $\in (2, 4)$

(D) Det (A) $\in [2, 4]$

Ans. Given: Matrix A = $\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

$\Rightarrow |A| = 1(1+\sin^2\theta) - \sin\theta(-\sin\theta + \sin\theta) + 1(\sin^2\theta + 1)$

$\Rightarrow |A| = 1+\sin^2\theta + 1 + \sin^2\theta = 2 + 2\sin^2\theta$ ....(i)

Since $-1 \le \sin\theta \le 1$

$\Rightarrow 0 \le \sin^2\theta \le 1$ [$\because \sin^2\theta$ cannot be negative]

$$\Rightarrow\ 0 \le 2\sin^2\theta \le 2$$

$$\Rightarrow\ 2 \le 2 + 2\sin^2\theta \le 4$$

$$\Rightarrow\ 2 \le \text{Det. A} \le 4$$

Therefore, option (D) is correct.