Chapter 13 Oscillations

EXERCISES PAGE: 274

1. Which of the following examples represents periodic motion?

(a) A swimmer completing one (return) trip from one bank of a river to the other and back.

(b) A freely suspended bar magnet displaced from its N-S direction and released.

(c) A hydrogen molecule rotating about its centre of mass.

(d) An arrow released from a bow.

Solution:

(a) The swimmers motion is not periodic. The motion of the swimmer between the banks of a river is to and fro. However, it does not have a definite period. This is because the time taken by the swimmer during his back and forth journey may not be the same.

(b) The motion of a freely-suspended magnet, if displaced from its N-S direction and released, is periodic. This is because the magnet oscillates about its position with a definite period of time.

(c) A hydrogen molecule rotating about its centre of mass is periodic. This is because when a hydrogen molecule rotates about its centre of mass, it comes to the same position again and again after an equal interval of time.

(d) An arrow released from a bow moves only in the forward direction. It does not come backward. Therefore, this motion is not periodic.

2. Which of the following examples represent (nearly) simple harmonic motion and which represent periodic but not simple harmonic motion?
(a) the rotation of earth about its axis.
(b) motion of an oscillating mercury column in a U-tube.
(c) motion of a ball bearing inside a smooth curved bowl, when released from a point slightly above the lower most point.

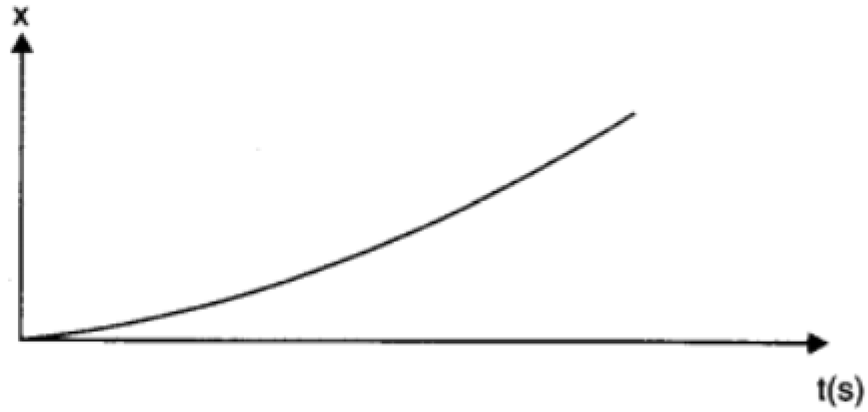
(d) general vibrations of a polyatomic molecule about its equilibrium position.

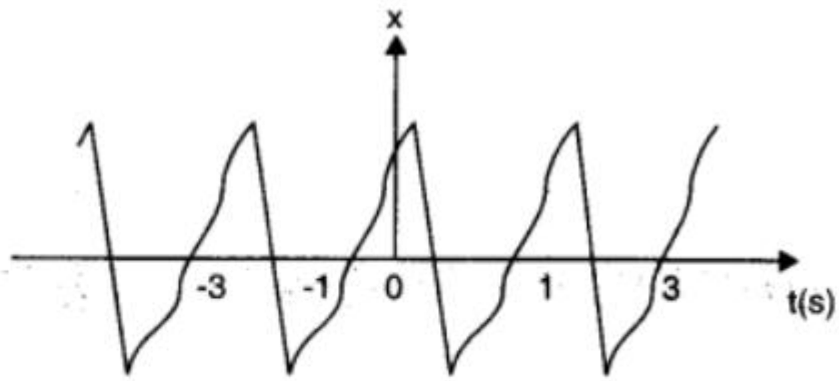
Solution:

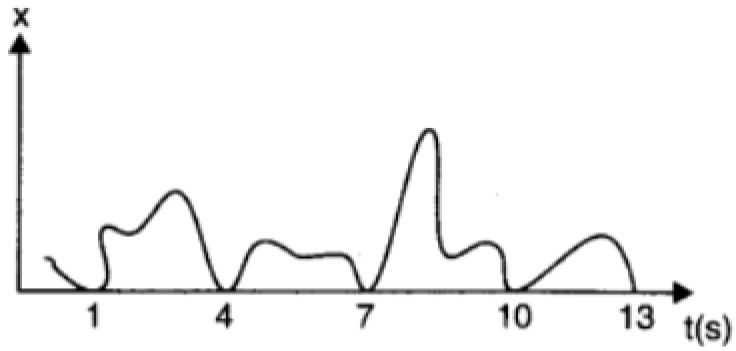
(a) Rotation of the earth is not to and fro motion about a fixed point. Therefore, it is periodic but not S.H.M.
(b) Simple harmonic motion
(c) Simple harmonic motion
(d) General vibrations of a polyatomic molecule about its equilibrium position is periodic but not SHM. A polyatomic molecule has a number of natural frequencies. Therefore, its vibration is a superposition of simple harmonic motions of a number of different frequencies.

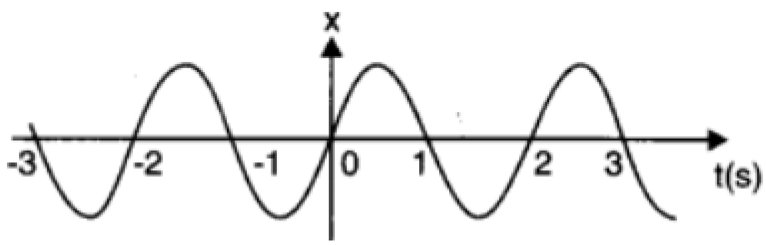
3. Given fig. depicts four x-t plots for linear motion of a particle. Which of the plots represent periodic motion? What is the period of motion (in

case of periodic motion)?









Solution:

(a) It is not periodic motion because motion does not repeat itself after a regular interval of time

(b) The given graph illustrates a periodic motion, which is repeating itself after every 2 seconds

(c) The given graph does not exhibit a periodic motion because the motion is repeated in one position only. For a periodic motion, the entire motion during one period must be repeated successively

(d) The given graph illustrates a periodic motion, which is repeating itself in every 2 seconds

4. Which of the following functions of time represent (a) simple harmonic, (b) periodic but not simple harmonic, and (c) non-periodic motion? Give period for each case of
periodic motion (ω is any positive constant):

(a) sin ωt – cos ωt

(b) $\sin^3 \omega t$

(c) 3 cos (π/4 – 2ωt)

(d) cos ωt + cos 3ωt + cos 5ωt

(e) $\exp(-\omega^2 t^2)$

(f) $1 + \omega t + \omega^2 t^2$

Solution:

(A)sin ωt – cos ωt

$$= \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin\omega t - \frac{1}{\sqrt{2}}\cos\omega t\right]$$
$$= \sqrt{2}\left[\sin\omega t \times \cos\frac{\pi}{4} - \cos\omega t \times \sin\frac{\pi}{4}\right]$$
$$= \sqrt{2}\sin\left(\omega t - \frac{\pi}{4}\right).$$

This function represents SHM as it can be written in the form : $a\ \sin(\omega t + \phi)$

Its period is $\frac{2\pi}{\omega}$.

(b) **$\sin^3 \omega t$**

The given function is:

$\sin^3 \omega t$

$\sin^3 \omega t = 1/4\ [3 \sin \omega t - \sin 3\omega t]$

The terms sin ωt and sin ωt individually represent simple harmonic motion (SHM). However, the superposition of two SHM is periodic and not simple harmonic.

Its period is: 2π/ω

(c) **3 cos (π/4 – 2ωt)**

The given function is: 3 cos (π/4 – 2ωt)

This function represents simple harmonic motion because it can be written in the form: a cos (ωt + Φ)Its period is: 2π/2ω = π/ω

(d) **cos ωt + cos 3ωt + cos 5ωt**

The given function is cosωt + cos3ωt + cos5ωt. Each individual cosine function represents SHM. However, the superposition of three simple harmonic motions is periodic, but not simple harmonic.

(e) $\exp(-\omega^2 t^2)$

It is an exponential function that does not repeat itself. Therefore, it is a non-periodic motion.

(f) The given function $1+\omega t+\omega^2 t^2$

It is non-periodic.

5. A particle is in linear simple harmonic motion between two points, A and B, 10 cm apart. Take the direction from A to B as the positive direction and give the signs of velocity, acceleration and force on the particle when it is

(a) at the end A

(b) at the end B

(c) at the mid-point of AB going towards A

(d) at 2 cm away from B going towards A

(e) at 3 cm away from A going towards B

(f) at 4 cm away from B going towards A

Solution:

(a) Zero, Positive, Positive

(b) Zero, Negative, Negative

(c) Negative, Zero, Zero

(d) Negative, Negative, Negative

(e) Positive, Positive, Positive

(f) Negative, Negative, Negative

Explanation:

(a), (b) The given situation is shown in the following figure. Points A and B are the two end points, with AB = 10 cm, where 'O' is the midpoint of the path.

A particle is in linear simple harmonic motion between the end points. At the extreme point A, the particle is at rest momentarily. Therefore, its velocity is zero at this point. Its acceleration is positive as it is directed along AO. Force is also positive in this case as the particle is directed rightward.

At the extreme point B, the particle is at rest momentarily. Therefore, its velocity is zero at this point

(c) The particle is executing a simple harmonic motion. 'O' is the mean position of the particle. Its velocity at the mean position O is the maximum. The value for velocity is negative since the particle is directed leftward. The acceleration and force of a particle executing SHM is zero at the mean position.

(d) The particle is moving toward point O from the end B. This direction of motion is opposite to the conventional positive direction, which is from A and B. Therefore, the particle's velocity and acceleration, and the force on it are all negative.

(e) The particle is moving toward point O from the end A. This direction of motion is from A to B, which is the conventional positive direction. Therefore, the values for velocity, acceleration and force are all positive.

(f) This case is similar to the one mentioned in (d).

6. Which of the following relationships between the acceleration a and the displacement x of a particle involve simple harmonic motion?

(a) a = 0.7x

(b) a = –200x²

(c) a = –10x

(d) a = 100x³

Solution:

Condition of SHM

Acceleration is directly proportional to negative of displacement of particle

If ‘a’ is acceleration

x is displacement

Then, for Simple Harmonic Motion,

a = – kx where k is constant

(a) a = 0.7x

This is not in the form of a = -kx

Hence, this is not SHM

(b) $a = -200x^2$

Clearly, it is not SHM

(c) $a = -10x$

This is in the form of $a = -kx$

Hence, this is SHM

(d) $a = 100x^3$

It's clear it is not SHM

7. The motion of a particle executing simple harmonic motion is described by the displacement function,
$x(t) = A \cos(\omega t + \varphi)$.
If the initial (t = 0) position of the particle is 1 cm and its initial velocity is ω cm/s, what are its amplitude and initial phase angle? The angular frequency of the particle is π s^{-1}. If instead of the cosine function, we choose the sine function to describe the SHM: $x = B \sin(\omega t + \alpha)$, what are the amplitude and initial phase of the particle with the above initial conditions.

Solution:

<u>Intially, at t = 0;</u>
Displacement, x = 1 cm
Intial velocity, v = ω cm/ sec.

Angular frequency, $\omega = \pi$ rad/s^{-1}

It is given that, $x(t) = A\cos(\omega t + \Phi)$ $1 = A\cos(\omega \times 0 + \Phi) = A\cos\Phi$ $A\cos\Phi =$

1 ...(i)

Velocity, $v = dx/dt$

$\omega = -A\,\omega\sin(\omega t + \Phi)$

$1 = -A\sin(\omega \times 0 + \Phi) = -A\sin\Phi$

$A\sin\Phi = -1$ ...(ii)

Squaring and adding equations (i) and (ii), we get:

$A^2(\sin^2\Phi + \cos^2\Phi) = 1 + 1$

$A^2 = 2$

$\therefore A = \sqrt{2}$ cm

Dividing equation (ii) by equation (i), we get:

$\tan\Phi = -1$

$\therefore \Phi = 3\pi/4$, $7\pi/4$,......

SHM is given as:

$x = B\sin(\omega t + \alpha)$

Putting the given values in this equation, we get:

$1 = B\sin[\omega \times 0 + \alpha] = 1 + 1$

$B\sin\alpha = 1$ ...(iii)

Velocity, $v = \omega B\cos(\omega t + \alpha)$

Substituting the given values, we get:

$\pi = \pi B\sin\alpha$

$B\sin\alpha = 1$ ...(iv)

Squaring and adding equations (iii) and (iv), we get:

$B^2[\sin^2\alpha + \cos^2\alpha] = 1 + 1$

$B^2 = 2$

$\therefore B = \sqrt{2}$ cm

Dividing equation (iii) by equation (iv), we get:

$B\sin\alpha / B\cos\alpha = 1/1$

$\tan\alpha = 1 = \tan\pi/4$

$\therefore \alpha = \pi/4$, $5\pi/4$,......

8. A spring balance has a scale that reads from 0 to 50 kg. The length of the scale is 20 cm. A body suspended from this balance, when displaced and released, oscillates with a period of 0.6 s. What is the weight of the body?

Solution:

Maximum mass that the scale can read, M = 50 kg

Maximum displacement of the spring = Length of the scale, l = 20 cm = 0.2 m

Time period, T = 0.6 s

Maximum force exerted on the spring, F = Mg

where,

g = acceleration due to gravity = 9.8 m/s^2

F = 50 × 9.8 = 490

∴ Spring constant, k = F/l = 490/0.2 = 2450 N m^{-1}.

Mass m, is suspended from the balance.

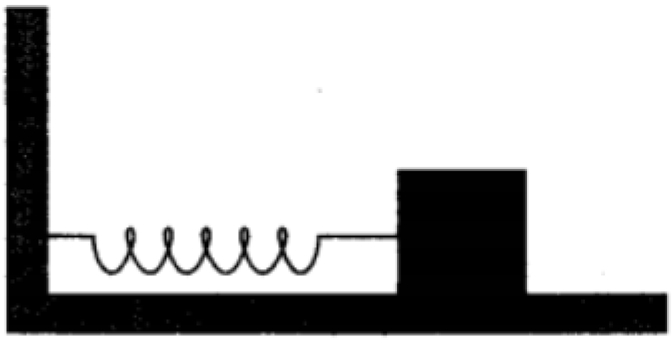
$$\text{Times period, } t = 2\pi\sqrt{\frac{m}{k}}$$

$$\therefore m = \left(\frac{T}{2\pi}\right)^2 \times k = \left(\frac{0.6}{2 \times 3.14}\right)^2 \times 2450 = 22.36\text{kg}$$

∴Weight of the body = mg = 22.36 × 9.8 = 219.167 N

Hence, the weight of the body is about 219 N.

9. A spring having with a spring constant 1200 N m^{-1} is mounted on a horizontal table as shown in Fig. 13.19. A mass of 3 kg is attached to the free end of the spring. The mass is then pulled sideways to a distance of 2.0 cm and released.



Determine (i) the frequency of oscillations, (ii) maximum acceleration of the mass, and (iii) the maximum speed of the mass.

Solution:

Here, $K = 1200\ Nm^{-1};\ m = 3.0\ kg,\ a = 2.0\ cm = 0.02\ m$

(i) Frequency, $\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\times 3.14}\sqrt{\frac{1200}{3}} = 3.2\ s^{-1}$

(ii) Acceleration, $A = \omega^2\ y = \frac{k}{m}y$

Acceleration will be maximum when y is maximum *i.e.*, $y = a$

$\therefore$ max. acceleration, $A_{max} = \frac{ka}{m} = \frac{1200\times 0.02}{3} = 8\ ms^{-2}$

(iii) Max. speed of the mass will be when it is passing through mean position

$$V_{max} = a\omega = a\sqrt{\frac{k}{m}} = 0.02 \times \sqrt{\frac{1200}{3}} = 0.4\ ms^{-1}$$

10. In Exercise 9, let us take the position of mass when the spring is unstreched as x = 0, and the direction from left to right as the positive direction of x-axis. Give x as a function of time t for the oscillating mass if at the moment we start the stopwatch (t = 0), the mass is

(a) at the mean position,

(b) at the maximum stretched position, and

(c) at the maximum compressed position. In what way do these functions for SHM differ from each other, in frequency, in amplitude or the initial phase?

Solution:

Distance travelled by the mass sideways, a = 2.0 cm

Angular frequency of oscillation:

$$a = 2cm, \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1200}{3}} = 20rads^{-1}$$

(a) When mass is at mean position
Then x =a sin ωt
x = 2 sin 20 t

(b) At maximum stretched position the mass is at extreme right position, hence, initial phase is $\frac{\pi}{2}$

$$\therefore x = a\sin\left(\omega t + \frac{\pi}{2}\right)$$
$$= 2\cos\omega t = 2\cos 20t$$

(c) At maximum compressed position, the mass is at extreme left with initial phase of $\frac{3\pi}{2}$ rad.

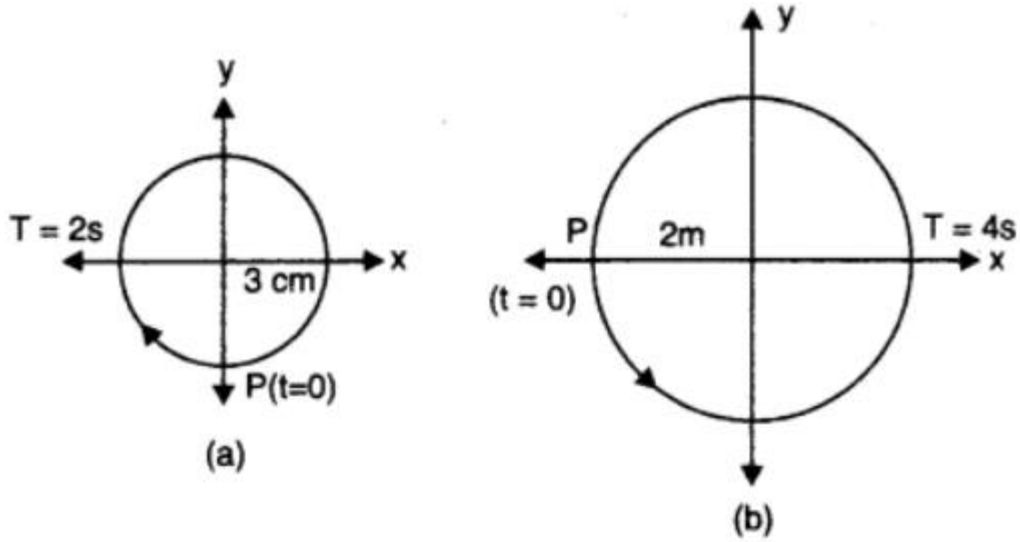
$$\therefore x = a\sin\left(\omega t + \frac{\pi}{2}\right)$$
$$= 2\cos\omega t = 2\cos 20t$$

(c) At maximum compressed position, the mass is at extreme left with initial phase of $\frac{3\pi}{2}$ rad.

$$\therefore x = a\sin\left(\omega t + \frac{3\pi}{2}\right)$$
$$= -a\cos\omega t = -2\cos 20t$$

The functions neither differ in amplitude nor in frequency. They differ in intial phase.

11. Figures 13.20 correspond to two circular motions. The radius of the circle, the period of revolution, the initial position, and the sense of revolution (i.e. clockwise or anti-clockwise) are indicated on each figure



Obtain the corresponding simple harmonic motions of the x-projection of the radius vector of the revolving particle P, in each case.

Solution:

(a) Time period, t = 2 s

Amplitude, A = 3 cm

At time, t = 0, the radius vector OP makes an angle π/2 with the positive x-axis,.e., phase angle Φ = +π/2

Therefore, the equation of simple harmonic motion for the x-projection of OP, at the time t, is given by the displacement equation:

$$x = A\cos\left[\frac{2\pi t}{T} + \phi\right]$$
$$= 3\cos\left(\frac{2\pi t}{2} + \frac{\pi}{2}\right) = -3\sin\left(\frac{2\pi t}{2}\right)$$
$$\therefore x = -3\sin(\pi t)\ \text{cm}$$

(b) Time Period, t = 4 s

Amplitude, a = 2 m

At time t = 0, OP makes an angle π with the x-axis, in the anticlockwise direction,

Hence, phase angle Φ = +π

Therefore, the equation of simple harmonic motion for the x-projection of OP, at the time t, is given as:

$$x = a\cos\left[\frac{2\pi t}{T} + \phi\right]$$
$$= 2\cos\left(\frac{2\pi t}{4} + \pi\right)$$
$$\therefore x = -2\cos\left(\frac{\pi}{2}t\right)\ m$$

12. Plot the corresponding reference circle for each of the following simple harmonic motions. Indicate the initial (t =0) position of the particle, the radius of the circle, and the angular speed of the rotating particle. For simplicity, the sense of rotation may be fixed to be anticlockwise in every case: (x is in cm and t is in s).

(a) x = –2 sin (3t + π/3)

(b) x = cos (π/6 – t)

(c) x = 3 sin (2πt + π/4)

(d) x = 2 cos πt

Solution:

a.

$$x = -2\sin\left(3t + \frac{\pi}{3}\right) = +2\cos\left(3t + \frac{\pi}{3} + \frac{\pi}{2}\right)$$
$$= 2\cos\left(3t + \frac{5\pi}{6}\right)$$

If this equation is compared with the standard SHM equation

$x = A\cos\left(\frac{2\pi}{T}t + \phi\right)$, **then we got,**

Amplitude, A = 2 cm

Phase angle, Φ = 5π/6 = 150°.

Angular velocity = ω = 2π/T = 3rad/sec.
The motion of the particle can be plotted as shown in fig. 10(a).
b.

$$x = \cos\left(\frac{\pi}{6} - t\right)$$
$$= \cos\left(t - \frac{\pi}{6}\right)$$

If this equation is compared with the standard SHM equation

$x = A\cos\left(\frac{2\pi}{T}t + \phi\right)$, than we got,

Amplitude, A = 1
Phase angle, Φ = -π/6 = -30º.
Angular velocity, ω = 2π/T = 1 rad/s.
The motion of the particle can be plotted as shown in fig. 10(b).

c.

$$x = 3\sin\left(2\pi t + \frac{\pi}{4}\right)$$
$$= -3\cos\left[\left(2\pi t + \frac{\pi}{4}\right) + \frac{\pi}{2}\right] = -3\cos\left(2\pi t + \frac{3\pi}{4}\right)$$

If this equation is compared with the standard SHM equation

$x = A\cos\left(\frac{2\pi}{T}t + \phi\right)$,

Amplitude, A = 3 cm
Phase angle, Φ = 3π/4 = 135º
Angular velocity, ω = 2π/T = 2 rad/s.
The motion of the particle can be plotted as shown in fig. 10(c).
d.

x = 2 cos πt

If this equation is compared with the standard SHM equation

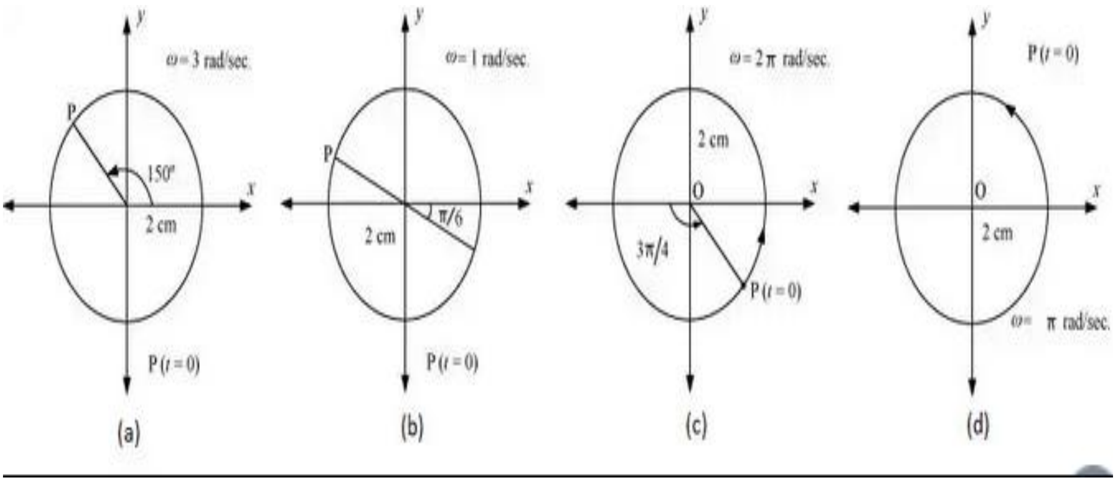
$$x = A\cos\left(\frac{2\pi}{T}t + \phi\right),$$

Amplitude, A = 2 cm

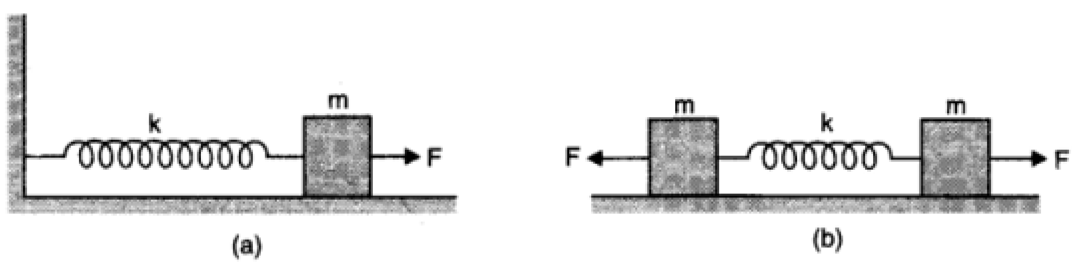
Phase angle, Φ = 0

Angular velocity, ω = π rad/s.

The motion of the particle can be plotted as shown in fig. 10(d).



13. Figure (a) shows a spring of force constant k clamped rigidly at one end and a mass m attached to its free end. A force F applied at the free end stretches the spring. Figure (b) shows the same spring with both ends free and attached to a mass m at either end. Each end of the spring in Fig. (b) is stretched by the same force F.



(a) What is the maximum extension of the spring in the two cases?

(b) If the mass in Fig. (a) and the two masses in Fig. (b) is released, what is the period of oscillation in each case?

Solution:

(*a*) Let y be the maximum extension produced in the spring in Fig. (*a*)

Then $F = ky$ (in magnitude) $\therefore \quad y = \frac{F}{k}$

If fig. (*b*), the force on one mass acts as the force of reaction due to the force on the other mass. Therefore, each mass behaves as if it is fixed with respect to the other.

Therefore, $F = ky \Rightarrow y = \frac{F}{k}$

(*b*) In fig. (*a*), $F = -ky$

$\Rightarrow \quad ma = -ky \Rightarrow a = -\frac{k}{m}y \quad \therefore \quad \omega^2 = \frac{k}{m} \quad$ *i.e.,* $\omega = \sqrt{\frac{k}{m}}$

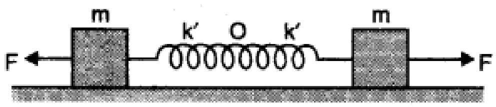
Therefore, period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

In fig. (*b*), we may consider that the centre of the system is O and there are two springs each of length $\frac{l}{2}$ attached to the two masses, each m, so that k' is the spring factor of each of the springs.

Then, $K' = 2k$

$\therefore \quad T = 2\pi\sqrt{\frac{m}{k'}}$

$= 2\pi\sqrt{\frac{m}{2k}}$



14. The piston in the cylinder head of a locomotive has a stroke (twice the amplitude) of 1.0 m. If the piston moves with simple harmonic motion with an angular frequency of 200 rad/min, what is its maximum speed?

Solution:

Stroke of piston = 2 times the amplitude

Let A = amplitude, stroke = 1 m

$\therefore \quad \Rightarrow A = \frac{1}{2}$ m.

Angular frequency, ω = 200 rad/min.

$$V_{max} = ?$$

We know that the maximum speed of the block when the amplitude is A,

$$V_{max} = \omega A = 200 \times \frac{1}{2} = 100 \text{ m/min.}$$

$$= \frac{100}{60} = \frac{5}{3} \text{ms}^{-1} = 1.67 \text{ ms}^{-1}.$$

15. The acceleration due to gravity on the surface of moon is 1.7 m s^{-2}. What is the time period of a simple pendulum on the surface of moon if its time period on the surface of earth is 3.5 s? (g on the surface of earth is 9.8 m s^{-2})

Solution:

Here, $g_m = 1.7 \text{ ms}^{-2}; \quad g_e = 9.8 \text{ ms}^{-2}; \quad T_m = ?; \quad T_e = 3.5 \text{ s}^{-1}$

Since, $T_e = 2\pi\sqrt{\frac{1}{g_e}}$ and $T_m = 2\pi\sqrt{\frac{1}{g_m}}$

$\therefore \quad \frac{T_m}{T_e} = \sqrt{\frac{g_e}{g_m}} \Rightarrow T_m = T_e = \sqrt{\frac{g_e}{g_m}}$

$$= 3.5\sqrt{\frac{9.8}{1.7}} = 8.4 \text{ s.}$$

16. A simple pendulum of length l and having a bob of mass M is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v. If the pendulum makes small oscillations in a radial direction about its equilibrium position, what will be its time period?

Solution: In this case, the bob of the pendulum is under the action of two accelerations.

(i) Acceleration due to gravity 'g' acting vertically downwards.

(ii) Centripetal acceleration $a_c = \frac{v^2}{R}$ acting along the horizontal direction.

$\therefore$ Effective acceleration, $g' = \sqrt{g^2 + a_c^2}$

or $$g' = \sqrt{g^2 + \frac{v^4}{R^2}}$$

Now time period, $T' = 2\pi\sqrt{\frac{1}{g}} = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + \frac{v^4}{R^2}}}}$.

17. A cylindrical piece of cork of density of base area A and height h floats in a liquid of density ρ_1. The cork is depressed slightly and then released. Show that the cork oscillates up and down simple harmonically with a period T = 2π √hρ / ρ₁g where ρ is the density of cork. (Ignore damping due to viscosity of the liquid)

Solution:

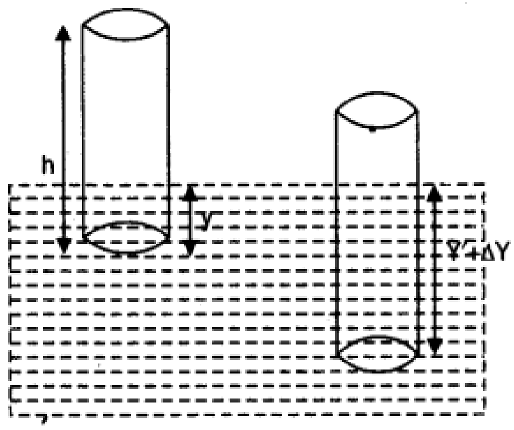
Say, initially in equilibrium, y height of cylinder is inside the liquid. Then,
Weight of the cylinder = upthrust due to liquid displaced

$$\therefore \quad Ah\rho g = Ay\rho_l g$$

When the cork cylinder is depressed slightly by Δy and released, a restoring force, equal to additional upthrust, acts on it. The restoring force is

$$F = A(y + \Delta y)\rho_l g - Ay\rho_l g = A\rho_l g\Delta y$$

$$\therefore \quad \text{Acceleration, } a = \frac{F}{m} = \frac{A\rho_l g\Delta y}{Ah\rho} = \frac{\rho_l g}{h\rho}.\Delta y$$ and the acceleration is directed in a direction opposite to Δy: Obviously, as $a \propto -\Delta y$, the motion of cork cylinder is SHM, whose time period is given by



$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}} = 2\pi\sqrt{\frac{\Delta y}{a}} = 2\pi\sqrt{\frac{h\rho}{\rho_l g}}.$$

18. One end of a U-tube containing mercury is connected to a suction pump and the other end to atmosphere. A small pressure difference is maintained between the two columns. Show that, when the suction pump is removed, the column of mercury in the U-tube executes simple harmonic motion.

Solution:

The suction pump creates the pressure difference, thus mercury rises in one limb of the U-tube. When it is removed, a net force acts on the liquid column due to the difference in levels of mercury in the two limbs and hence the liquid column executes S.H.M. which can be expressed as:
Consider the mercury contained in a vertical U-tube upto the level P and Q in its two limbs.
Let P = density of the mercury.
L = Total length of the mercury column in both the limbs.

A = internal cross-sectional area of U-tube. m = mass of mercury in U-tube = LAP.

Assume, the mercury be depressed in left limb to F by a small distance y, then it rises by the same amount in the right limb to position Q'.

.'. Difference in levels in the two limbs = P' Q' = 2y.

:. Volume of mercury contained in the column of length 2y = A X 2y

.•. m – A x 2y x ρ.

If W = weight of liquid contained in the column of length 2y.

Then W = mg = A x 2y x ρ x g

This weight produces the restoring force (F) which tends to bring back the mercury to its equilibrium position.

∴ $F = -2Ay\rho g = -(2A\rho g)y$

If a = acceleration produced in the liquid column, Then

$$a = \frac{F}{m}$$

$$= -\frac{(2A\rho g)y}{LA\rho} = -\frac{2A\rho g}{LA}$$

$$= -\frac{2\rho g}{2h\rho}y \qquad \text{...}(i) \quad (\because L = 2h)$$

where h = height of mercury in each limb. Now from eqn. (i), it is clear that $a \propto y$ and $-ve$ sign shows that it acts opposite to y, so the motion of mercury in u-tube is simple harmonic in nature having time period (T) given by

$$T = 2\pi\sqrt{\frac{y}{a}} = 2\pi\sqrt{\frac{2h\rho}{2\rho g}} = 2\pi\sqrt{\frac{h\rho}{\rho g}}$$

$$T = 2\pi\sqrt{\frac{h}{g}}$$