Exercise 7.1 Page: 234

Find an anti derivative (or integral) of the following functions by the method of inspection.

Question 1 sin 2*x*

Solution :

The anti derivative of sin 2*x* is a function of *x* whose derivative is sin 2*x*.

It is known that,

$$\frac{d}{dx}(\cos 2x) = -2\sin 2x$$
$$\Rightarrow \sin 2x = -\frac{1}{2}\frac{d}{dx}(\cos 2x)$$
$$\therefore \sin 2x = \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$$

herefore, the anti derivative of sin 2x is -1/2 cos 2x.

Question 2. cos 3*x*

Solution :

The anti derivative of cos 3*x* is a function of *x* whose derivative is cos 3*x*.

It is known that,

$$\frac{d}{dx}(\sin 3x) = 3\cos 3x$$
$$\Rightarrow \cos 3x = \frac{1}{3}\frac{d}{dx}(\sin 3x)$$
$$\therefore \cos 3x = \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)$$

Therefore, the anti derivative of cos 3x is 1/3 sin 3x.

Question 3. e^{2x}

Solution :

The anti derivative of e^{2x} is the function of x whose derivative is e^{2x}.

It is known that,

$$\frac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$$
$$\Rightarrow e^{2x} = \frac{1}{2}\frac{d}{dx}\left(e^{2x}\right)$$
$$\therefore e^{2x} = \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

Therefore, the anti derivative of e^{2x} is 1/2 e^{2x}.

Question 4. $(ax + b)^2$

Solution :

The anti derivative of $(ax + b)^2$ is the function of x whose derivative is $(ax + b)^2$.

It is known that,

$$\frac{d}{dx}(ax+b)^3 = 3a(ax+b)^2$$
$$\Rightarrow (ax+b)^2 = \frac{1}{3a}\frac{d}{dx}(ax+b)^3$$
$$\therefore (ax+b)^2 = \frac{d}{dx}\left(\frac{1}{3a}(ax+b)^3\right)$$

Therefore, the anti derivative of $(ax+b)^2$ is $\frac{1}{3a}(ax+b)^3$.

Question 5. $\sin 2x - 4e^{3x}$

Solution :

The anti derivative of **$\sin 2x - 4e^{3x}$** is the function of x whose derivative is **$\sin 2x - 4e^{3x}$**

It is known that,

$$\frac{d}{dx}\left(-\frac{1}{2}\cos 2x-\frac{4}{3}e^{3x}\right)=\sin 2x-4e^{3x}$$

Therefore, the anti derivative of $\left(\sin 2x-4e^{3x}\right)$ is $\left(-\frac{1}{2}\cos 2x-\frac{4}{3}e^{3x}\right)$.

Evaluate the following integrals in Exercises 6 to 11.
Question 6. $\int(4e^{3x}+1)\,dx$

Solution

$$\int\left(4e^{3x}+1\right)dx$$

$$=4\int e^{3x}dx+\int 1dx$$
$$=4\left(\frac{e^{3x}}{3}\right)+x+C$$
$$=\frac{4}{3}e^{3x}+x+C$$

Question 7. $\int x^2\left(1-\frac{1}{x^2}\right)dx$

Solution :

$$\int x^2\left(1-\frac{1}{x^2}\right)dx$$

$$=\int\left(x^2-1\right)dx$$
$$=\int x^2dx-\int 1dx$$
$$=\frac{x^3}{3}-x+C$$

Question 8. $\int\left(ax^2+bx+c\right)dx$

Solution

$$\int\left(ax^2+bx+c\right)dx$$

$$= a\int x^2dx + b\int xdx + c\int 1.dx$$

$$= a\left(\frac{x^3}{3}\right) + b\left(\frac{x^2}{2}\right) + cx + C$$

$$= \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C$$

Question 9. $\int\left(2x^2+e^x\right)dx$

Solution :

$$\int\left(2x^2+e^x\right)dx$$

$$= 2\int x^2dx + \int e^x dx$$

$$= 2\left(\frac{x^3}{3}\right) + e^x + C$$

$$= \frac{2}{3}x^3 + e^x + C$$

Question 10. $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx$

Solution :

$$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx$$

$$= \int\left(x+\frac{1}{x}-2\right)dx$$

$$= \int xdx + \int\frac{1}{x}dx - 2\int 1.dx$$

$$= \frac{x^2}{2} + \log|x| - 2x + C$$

Question 11. $\int\frac{x^3+5x^2-4}{x^2}dx$

Solution :

$$\int \frac{x^3+5x^2-4}{x^2}dx$$

$$= \int\left(x+5-4x^{-2}\right)dx$$
$$= \int x dx + 5\int 1.dx - 4\int x^{-2}dx$$
$$= \frac{x^2}{2}+5x-4\left(\frac{x^{-1}}{-1}\right)+C$$
$$= \frac{x^2}{2}+5x+\frac{4}{x}+C$$

Evaluate the following integrals in Exercises 12 to 16.

Question 12. $\int \frac{x^3+3x+4}{\sqrt{x}}dx$

Solution :

$$= \int\left(x^{\frac{5}{2}}+3x^{\frac{1}{2}}+4x^{-\frac{1}{2}}\right)dx$$
$$= \frac{x^{\frac{7}{2}}}{\frac{7}{2}}+\frac{3\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}}+\frac{4\left(x^{\frac{1}{2}}\right)}{\frac{1}{2}}+C$$
$$= \frac{2}{7}x^{\frac{7}{2}}+2x^{\frac{3}{2}}+8x^{\frac{1}{2}}+C$$
$$= \frac{2}{7}x^{\frac{7}{2}}+2x^{\frac{3}{2}}+8\sqrt{x}+C$$

Question 13. $\int \frac{x^3-x^2+x-1}{x-1}dx$

Solution :

$$\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$$

On dividing, we obtain

$$= \int (x^2 + 1) dx$$
$$= \int x^2 dx + \int 1 dx$$
$$= \frac{x^3}{3} + x + C$$

Question 14. ∫(1 – x)√x dx

Solution : ∫(1 – x)√x dx

$$= \int \left(\sqrt{x} - x^{\frac{3}{2}} \right) dx$$
$$= \int x^{\frac{1}{2}} dx - \int x^{\frac{3}{2}} dx$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C$$
$$= \frac{2}{3} x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}} + C$$

Question 15. $\int \sqrt{x} \left(3x^2 + 2x + 3 \right) dx$

Solution :

$$\int\sqrt{x}\left(3x^2+2x+3\right)dx$$

$$= \int\left(3x^{\frac{5}{2}}+2x^{\frac{3}{2}}+3x^{\frac{1}{2}}\right)dx$$

$$= 3\int x^{\frac{5}{2}}dx+2\int x^{\frac{3}{2}}dx+3\int x^{\frac{1}{2}}dx$$

$$= 3\left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}}\right)+2\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)+3\frac{\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}}+C$$

$$= \frac{6}{7}x^{\frac{7}{2}}+\frac{4}{5}x^{\frac{5}{2}}+2x^{\frac{3}{2}}+C$$

Question 16. $\int\left(2x-3\cos x+e^x\right)dx$

Solution

$$\int\left(2x-3\cos x+e^x\right)dx$$

$$= 2\int xdx-3\int\cos xdx+\int e^x dx$$

$$= \frac{2x^2}{2}-3\left(\sin x\right)+e^x+C$$

$$= x^2-3\sin x+e^x+C$$

Evaluate the following integrals in Exercises 17 to 20.

Question 17. $\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx$

Solution

$$\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx$$

$$= 2\int x^2dx-3\int\sin xdx+5\int x^{\frac{1}{2}}dx$$

$$= \frac{2x^3}{3}-3\left(-\cos x\right)+5\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)+C$$

$$= \frac{2}{3}x^3+3\cos x+\frac{10}{3}x^{\frac{3}{2}}+C$$

Question 18. $\int\sec x\left(\sec x+\tan x\right)dx$

Solution

$\int \sec x(\sec x + \tan x)\,dx$

$= \int (\sec^2 x + \sec x \tan x)\,dx$

$= \int \sec^2 x dx + \int \sec x \tan x dx$

$= \tan x + \sec x + C$

Question 19. $\int \frac{\sec^2 x}{\cos ec^2 x}\,dx$

Solution :

$\int \frac{\sec^2 x}{\cos ec^2 x}\,dx$

$= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}}\,dx$

$= \int \frac{\sin^2 x}{\cos^2 x}\,dx$

$= \int \tan^2 x dx$

$= \int (\sec^2 x - 1)\,dx$

$= \int \sec^2 x dx - \int 1 dx$

$= \tan x - x + C$

Question 20. $\int \frac{2 - 3\sin x}{\cos^2 x}\,dx$

Solution

$\int \frac{2 - 3\sin x}{\cos^2 x}\,dx$

$= \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x} \right) dx$

$= \int 2\sec^2 x dx - 3\int \tan x \sec x dx$

$= 2\tan x - 3\sec x + C$

Question 21. Choose the correct answer:

The anti derivative of $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$ **equals.**

(A) $\frac{1}{3}x^{\frac{1}{3}}+2x^{\frac{1}{2}}+C$

(B) $\frac{2}{3}x^{\frac{2}{3}}+\frac{1}{2}x^{2}+C$

(C) $\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+C$

(D) $\frac{3}{2}x^{\frac{3}{2}}+\frac{1}{2}x^{\frac{1}{2}}+C$

Solution

$$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$$

$$=\int x^{\frac{1}{2}}dx+\int x^{-\frac{1}{2}}dx$$

$$=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+C$$

$$=\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+C$$

Therefore, option (C) is correct.

Question 22. Choose the correct answer:

If $\frac{d}{dx}f(x)=4x^{3}-\frac{3}{x^{4}}$ **such that f(2) = 0 Then f is:**

(A) $x^{4}+\frac{1}{x^{3}}-129$

(B)$x^{3}+\frac{1}{x^{4}}+\frac{129}{8}$

(C) $x^{4}+\frac{1}{x^{3}}+\frac{129}{8}$

(D) $x^{3}+\frac{1}{x^{4}}-\frac{129}{8}$

Solution :

It is given that,

$$\frac{d}{dx}f(x)=4x^3-\frac{3}{x^4}$$

$\therefore$Anti derivative of $4x^3-\frac{3}{x^4}=f(x)$

$$\therefore f(x)=\int 4x^3-\frac{3}{x^4}\,dx$$

$$f(x)=4\int x^3dx-3\int\left(x^{-4}\right)dx$$

$$\therefore\ f(x)=4\left(\frac{x^4}{4}\right)-3\left(\frac{x^{-3}}{-3}\right)+C$$

$$f(x)=x^4+\frac{1}{x^3}+C$$

Also,

$$f(2)=0$$

$$\therefore f(2)=(2)^4+\frac{1}{(2)^3}+C=0$$

$$\Rightarrow 16+\frac{1}{8}+C=0$$

$$\Rightarrow C=-\left(16+\frac{1}{8}\right)$$

$$\Rightarrow C=\frac{-129}{8}$$

$$\therefore f(x)=x^4+\frac{1}{x^3}-\frac{129}{8}$$

Therefore, option (A) is correct.

Exercise 7.2 Page: 240

Integrate the functions in Exercise 1 to 8.

Question 1. $\frac{2x}{1+x^2}$

Solution :

Let $1 + x^2 = t$

$\therefore 2x \, dx = dt$

$$\Rightarrow \int \frac{2x}{1+x^2} dx = \int \frac{1}{t} dt$$

$$= \log|t| + C$$
$$= \log\left|1+x^2\right| + C$$
$$= \log\left(1+x^2\right) + C$$

Question 2. $\frac{(\log x)^2}{x}$

Solution :

Let $\log |x| = t$

$\therefore 1/x dx = dt$

$$\Rightarrow \int \frac{(\log|x|)^2}{x} dx = \int t^2 dt$$
$$= \frac{t^3}{3} + C$$
$$= \frac{(\log|x|)^3}{3} + C$$

Question 3. $\frac{1}{x + x\log x}$

Solution :

$$\frac{1}{x + x\log x} = \frac{1}{x(1+\log x)}$$

Let $1 + \log x = t$

$$\therefore \frac{1}{x} dx = dt$$

$$\Rightarrow \int \frac{1}{x(1+\log x)} dx = \int \frac{1}{t} dt$$

$$= \log|t| + C$$
$$= \log|1+\log x| + C$$

Question 4. sin x ⋅ sin (cos x)

Solution :

sin x · sin (cos x)

Let cos x = t

$\therefore -\sin x dx = dt$

$$\Rightarrow \int \sin x \cdot \sin(\cos x)\,dx = -\int \sin t\,dt$$
$$= -[-\cos t] + C$$
$$= \cos t + C$$
$$= \cos(\cos x) + C$$

Question 5. sin(ax + b) cos(ax + b)

Solution :

$$\sin(ax+b)\cos(ax+b) = \frac{2\sin(ax+b)\cos(ax+b)}{2} = \frac{\sin 2(ax+b)}{2}$$

Let $2(ax+b) = t$

$\therefore 2adx = dt$

$$\Rightarrow \int \frac{\sin 2(ax+b)}{2}dx = \frac{1}{2}\int \frac{\sin t\ dt}{2a}$$
$$= \frac{1}{4a}[-\cos t] + C$$
$$= \frac{-1}{4a}\cos 2(ax+b) + C$$

Questionc 6. $\sqrt{ax + b}$

Solution :

Let $ax + b = t$

$\Rightarrow adx = dt$

$$\therefore dx = \frac{1}{a}dt$$

$$\Rightarrow \int (ax+b)^{\frac{1}{2}} dx = \frac{1}{a}\int t^{\frac{1}{2}} dt$$

$$= \frac{1}{a}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{3a}(ax+b)^{\frac{3}{2}} + C$$

Question 7. x√x + 2

Solution :

Let (x + 2) = t

$\therefore dx = dt$

$$\Rightarrow \int x\sqrt{x+2}\,dx = \int (t-2)\sqrt{t}\,dt$$

$$= \int \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right) dt$$

$$= \int t^{\frac{3}{2}} dt - 2\int t^{\frac{1}{2}} dt$$

$$= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$$

Question8. x√1+2x^2
Solution :

Let 1 + 2x^2 = t

$\therefore 4x\,dx = dt$

$$\Rightarrow \int x\sqrt{1+2x^2}\,dx = \int \frac{\sqrt{t}\,dt}{4}$$
$$= \frac{1}{4}\int t^{\frac{1}{2}}\,dt$$
$$= \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$
$$= \frac{1}{6}\left(1+2x^2\right)^{\frac{3}{2}} + C$$

Integrate the functions in Exercise 9 to 17.

Question 9. $(4x+2)\sqrt{x^2+x+1}$

Solution **:**

Let $x^2+x+1=t$

$\therefore (2x+1)dx = dt$

$$\int (4x+2)\sqrt{x^2+x+1}\;dx$$

$$= \int 2\sqrt{t}\;dt$$
$$= 2\int \sqrt{t}\;dt$$
$$= 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$
$$= \frac{4}{3}\left(x^2+x+1\right)^{\frac{3}{2}} + C$$

Question 10. $\dfrac{1}{x-\sqrt{x}}$

Solution **:**

$$\frac{1}{x-\sqrt{x}}=\frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}$$

Let $\left(\sqrt{x}-1\right)=t$

$$\therefore \frac{1}{2\sqrt{x}}dx=dt$$

$$\Rightarrow \int\frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}dx=\int\frac{2}{t}dt$$

$$=2\log|t|+C$$
$$=2\log\left|\sqrt{x}-1\right|+C$$

Question 11. $\frac{x}{\sqrt{x+4}}, x>0$

Solution **:**

$$\text{Let } I=\int\frac{x}{(x+4)}\,dx$$
$$\text{put } x+4=t$$
$$\Rightarrow dx=dt$$
$$\text{Now, } I=\int\frac{(t-4)}{\sqrt{t}}dt$$
$$=\int\left(\sqrt{t}-4t^{-1/2}\right)dt$$
$$=\frac{2}{3}t^{3/2}-4\left(2t^{1/2}\right)+C$$
$$=\frac{2}{3}.t.t^{1/2}-8t^{1/2}+C$$
$$=\frac{2}{3}(x+4)\sqrt{x+4}-8\sqrt{x+4}+C$$
$$=\frac{2}{3}x\sqrt{x+4}+\frac{8}{3}\sqrt{x+4}-8\sqrt{x+4}+C$$
$$=\frac{2}{3}x\sqrt{x+4}-\frac{16}{3}\sqrt{x+4}+C$$
$$=\frac{2}{3}\left(\sqrt{x+4}\right)(x-8)+C$$

Question 12. $\left(x^3-1\right)^{\frac{1}{3}}x^5$

Solution :

Let $x^3 - 1 = t$

$\therefore 3x^2\,dx = dt$

$$\Rightarrow \int \left(x^3-1\right)^{\frac{1}{3}} x^5 dx = \int \left(x^3-1\right)^{\frac{1}{3}} x^3 \cdot x^2 dx$$
$$= \int t^{\frac{1}{3}} (t+1) \frac{dt}{3}$$
$$= \frac{1}{3} \int \left(t^{\frac{4}{3}} + t^{\frac{1}{3}} \right) dt$$
$$= \frac{1}{3} \left[\frac{t^{\frac{7}{3}}}{\frac{7}{3}} + \frac{t^{\frac{4}{3}}}{\frac{4}{3}} \right] + C$$
$$= \frac{1}{3} \left[\frac{3}{7} t^{\frac{7}{3}} + \frac{3}{4} t^{\frac{4}{3}} \right] + C$$
$$= \frac{1}{7} \left(x^3-1\right)^{\frac{7}{3}} + \frac{1}{4} \left(x^3-1\right)^{\frac{4}{3}} + C$$

Question 13. $\frac{x^2}{\left(2+3x^3\right)^3}$

Solution :

Let $2 + 3x^3 = t$

$\therefore 9x^2\,dx = dt$

$$\Rightarrow \int \frac{x^2}{\left(2+3x^3\right)^3} dx = \frac{1}{9} \int \frac{dt}{(t)^3}$$
$$= \frac{1}{9} \left[\frac{t^{-2}}{-2} \right] + C$$
$$= \frac{-1}{18} \left(\frac{1}{t^2} \right) + C$$
$$= \frac{-1}{18\left(2+3x^3\right)^2} + C$$

Question14. $\frac{1}{x(\log x)^m}\ ,\ x > 0$

Solution :

Let log $x = t$

∴ $1/x \, dx = dt$

$$\Rightarrow \int \frac{1}{x(\log x)^m} dx = \int \frac{dt}{(t)^m}$$

$$= \left(\frac{t^{-m+1}}{1-m}\right) + C$$

$$= \frac{(\log x)^{1-m}}{(1-m)} + C$$

Question 15. $x/9 - 4x^2$

Solution :

Let $9 - 4x^2 = t$

∴ $-8x \, dx = dt$

$$\Rightarrow \int \frac{x}{9-4x^2} dx = \frac{-1}{8} \int \frac{1}{t} dt$$

$$= \frac{-1}{8} \log|t| + C$$

$$= \frac{-1}{8} \log\left|9-4x^2\right| + C$$

Question 16. e^{2x+3}

Solution :

Let $2x + 3 = t$

∴ $2dx = dt$

$$\Rightarrow \int e^{2x+3} dx = \frac{1}{2} \int e^t dt$$

$$= \frac{1}{2}(e^t) + C$$

$$= \frac{1}{2} e^{(2x+3)} + C$$

Question17. $\frac{x}{e^{x^2}}$
Solution :

Let $x^2 = t$

$\therefore 2xdx = dt$

$$\Rightarrow \int \frac{x}{e^{x^2}} dx = \frac{1}{2} \int \frac{1}{e^t} dt$$

$$= \frac{1}{2} \int e^{-t} dt$$
$$= \frac{1}{2}\left(\frac{e^{-t}}{-1}\right) + C$$
$$= -\frac{1}{2} e^{-x^2} + C$$
$$= \frac{-1}{2e^{x^2}} + C$$

Integrate the functions in Exercise 18 to 26.

Question 18. $\frac{e^{\tan^{-1} x}}{1+x^2}$

Solution **:**
Let $\tan^{-1} x = t$

$$\therefore \frac{1}{1+x^2} dx = dt$$

$$\Rightarrow \int \frac{e^{\tan^{-1} x}}{1+x^2} dx = \int e^t dt$$

$$= e^t + C$$
$$= e^{\tan^{-1} x} + C$$

Question 19. $\frac{e^{2x}-1}{e^{2x}+1}$

Solution **:**

$$\frac{e^{2x}-1}{e^{2x}+1}$$

Dividing numerator and denominator by e^x, we obtain

$$\frac{\frac{\left(e^{2x}-1\right)}{e^x}}{\frac{\left(e^{2x}+1\right)}{e^x}}=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

Let $e^x+e^{-x}=t$

$$\therefore \left(e^x-e^{-x}\right)dx=dt$$

$$\Rightarrow \int\frac{e^{2x}-1}{e^{2x}+1}dx=\int\frac{e^x-e^{-x}}{e^x+e^{-x}}dx$$

$$=\int\frac{dt}{t}$$
$$=\log|t|+C$$
$$=\log\left|e^x+e^{-x}\right|+C$$

Question20. $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$
Solution :

Let $e^{2x}+e^{-2x}=t$

$$\therefore \left(2e^{2x}-2e^{-2x}\right)dx=dt$$

$$\Rightarrow 2\left(e^{2x}-e^{-2x}\right)dx=dt$$

$$\Rightarrow \int\left(\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\right)dx=\int\frac{dt}{2t}$$
$$=\frac{1}{2}\int\frac{1}{t}dt$$
$$=\frac{1}{2}\log|t|+C$$
$$=\frac{1}{2}\log\left|e^{2x}+e^{-2x}\right|+C$$

Question 21. $\tan^2 (2x - 3)$

Solution :

$$\tan^2(2x-3) = \sec^2(2x-3) - 1$$

Let $2x - 3 = t$

$\therefore 2dx = dt$

$$\Rightarrow \int \tan^2 (2x - 3)dx$$
$$= \int [\sec^2 (2x - 3) - 1]dx$$
$$= \int [\sec^2 t - 1] \frac{dt}{2}$$
$$= \frac{1}{2} [\int \sec^2 t \, dt - \int 1dt]$$
$$= \frac{1}{2} [\tan t - t + C]$$
$$= \frac{1}{2} [\tan (2x - 3) - (2x - 3) + C]$$

Question 22. $\sec^2 (7 - 4x)$

Solution :

Let $7 - 4x = t$

$\therefore -4dx = dt$

$$\therefore \int \sec^2(7-4x)dx = \frac{-1}{4}\int \sec^2 t\, dt$$
$$= \frac{-1}{4}(\tan t) + C$$
$$= \frac{-1}{4}\tan(7-4x) + C$$

Question 23. $\dfrac{\sin^{-1} x}{\sqrt{1 - x^2}}$

Solution :

Let $\sin^{-1} x = t$

$\therefore \frac{1}{\sqrt{1-x^2}} dx = dt$

$\Rightarrow \int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int t\, dt$

$= \frac{t^2}{2} + C$

$= \frac{\left(\sin^{-1} x\right)^2}{2} + C$

Question 24. $\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$

Solution :

$\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} = \frac{2\cos x - 3\sin x}{2(3\cos x + 2\sin x)}$

Let $3\cos x + 2\sin x = t$

$\therefore (-3\sin x + 2\cos x) dx = dt$

$$\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} dx = \int \frac{dt}{2t}$$
$$= \frac{1}{2} \int \frac{1}{t} dt$$
$$= \frac{1}{2} \log|t| + C$$
$$= \frac{1}{2} \log|2\sin x + 3\cos x| + C$$

Question 25. $\frac{1}{\cos^{23} x(1-\tan x)^2}$

Solution **:**

$$\frac{1}{\cos^2 x(1-\tan x)^2} = \frac{\sec^2 x}{(1-\tan x)^2}$$

Let $(1-\tan x) = t$

$\therefore \; -\sec^2 x dx = dt$

$$\Rightarrow \int \frac{\sec^2 x}{(1-\tan x)^2} dx = \int \frac{-dt}{t^2}$$
$$= -\int t^{-2} dt$$
$$= +\frac{1}{t} + C$$
$$= \frac{1}{(1-\tan x)} + C$$

Question 26. $\dfrac{\cos\sqrt{x}}{\sqrt{x}}$

Solution :

Let √x = t

$\therefore \; \frac{1}{2\sqrt{x}} dx = dt$

$$\Rightarrow \int \frac{\cos\sqrt{x}}{\sqrt{x}} dx = 2\int \cos t \, dt$$
$$= 2\sin t + C$$
$$= 2\sin\sqrt{x} + C$$

Integrate the functions in Exercise 27 to 37.

Question 27. $\sqrt{\sin 2x} \cos 2x$

Solution :

Let sin 2*x* = *t*

$$\Rightarrow \int \sqrt{\sin 2x}\ \cos 2x\,dx = \frac{1}{2}\int \sqrt{t}\,dt$$
$$= \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$
$$= \frac{1}{3}t^{\frac{3}{2}} + C$$
$$= \frac{1}{3}(\sin 2x)^{\frac{3}{2}} + C$$

Question 28. $\dfrac{\cos x}{\sqrt{1+\sin x}}$

Solution :

Let 1 + sin x = t

∴ cos *x dx* = *dt*

$$\Rightarrow \int \frac{\cos x}{\sqrt{1+\sin x}}dx = \int \frac{dt}{\sqrt{t}}$$
$$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$
$$= 2\sqrt{t} + C$$
$$= 2\sqrt{1+\sin x} + C$$

Question 29. cot *x* log sin *x*

Solution :

Let log sin *x* = *t*

$$\Rightarrow \frac{1}{\sin x} \cdot \cos x \; dx = dt$$
$$\therefore \cot x \; dx = dt$$

$$\Rightarrow \int \cot x \; \log \sin x \; dx = \int t \, dt$$
$$= \frac{t^2}{2} + C$$
$$= \frac{1}{2} (\log \sin x)^2 + C$$

Question 30. sin x/1 + cos *x*

Solution :

Let 1 + cos $x = t$

$\therefore -\sin x \, dx = dt$

$$\Rightarrow \int \frac{\sin x}{1+\cos x} \, dx = \int -\frac{dt}{t}$$
$$= -\log|t| + C$$
$$= -\log|1+\cos x| + C$$

Question 31. sin x/(1 + cos *x*)²

Solution :

Let 1 + cos $x = t$

$\therefore -\sin x \, dx = dt$

$$\Rightarrow \int \frac{\sin x}{(1+\cos x)^2} \, dx = \int -\frac{dt}{t^2}$$
$$= -\int t^{-2} dt$$
$$= \frac{1}{t} + C$$
$$= \frac{1}{1+\cos x} + C$$

Question32.1/1+cot x
Solution :

$$\text{Let } I = \int \frac{1}{1+\cot x}dx$$
$$= \int \frac{1}{1+\frac{\cos x}{\sin x}}dx$$
$$= \int \frac{\sin x}{\sin x + \cos x}dx$$
$$= \frac{1}{2}\int \frac{2\sin x}{\sin x + \cos x}dx$$
$$= \frac{1}{2}\int \frac{(\sin x + \cos x)+(\sin x - \cos x)}{(\sin x + \cos x)}dx$$
$$= \frac{1}{2}\int 1\,dx + \frac{1}{2}\int \frac{\sin x - \cos x}{\sin x + \cos x}dx$$
$$= \frac{1}{2}(x) + \frac{1}{2}\int \frac{\sin x - \cos x}{\sin x + \cos x}dx$$

Let $\sin x + \cos x = t \Rightarrow (\cos x - \sin x)\, dx = dt$

$$\therefore I = \frac{x}{2} + \frac{1}{2}\int \frac{-(dt)}{t}$$
$$= \frac{x}{2} - \frac{1}{2}\log|t| + C$$
$$= \frac{x}{2} - \frac{1}{2}\log|\sin x + \cos x| + C$$

Question 33. 1/1 – tan x

Solution :

$$\text{Let } I = \int \frac{1}{1-\tan x}dx$$
$$= \int \frac{1}{1-\frac{\sin x}{\cos x}}dx$$
$$= \int \frac{\cos x}{\cos x - \sin x}dx$$
$$= \frac{1}{2}\int \frac{2\cos x}{\cos x - \sin x}dx$$
$$= \frac{1}{2}\int \frac{(\cos x - \sin x) + (\cos x + \sin x)}{(\cos x - \sin x)}dx$$
$$= \frac{1}{2}\int 1\,dx + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x - \sin x}dx$$
$$= \frac{x}{2} + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x - \sin x}dx$$

Put $\cos x - \sin x = t \Rightarrow (-\sin x - \cos x)\,dx = dt$

$$\therefore I = \frac{x}{2} + \frac{1}{2}\int \frac{-(dt)}{t}$$
$$= \frac{x}{2} - \frac{1}{2}\log|t| + C$$
$$= \frac{x}{2} - \frac{1}{2}\log|\cos x - \sin x| + C$$

Question 34. $\dfrac{\sqrt{\tan x}}{\sin x \cos x}$

Solution **:**

$$\text{Let } I = \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$$
$$= \int \frac{\sqrt{\tan x} \times \cos x}{\sin x \cos x \times \cos x} dx$$
$$= \int \frac{\sqrt{\tan x}}{\tan x \cos^2 x} dx$$
$$= \int \frac{\sec^2 x \, dx}{\sqrt{\tan x}}$$

$$\text{Let } \tan x = t \Rightarrow \sec^2 x \, dx = dt$$

$$\therefore I = \int \frac{dt}{\sqrt{t}}$$
$$= 2\sqrt{t} + C$$
$$= 2\sqrt{\tan x} + C$$

Question 35. $\frac{(1+\log x)^2}{x}$

Solution :

Let 1 + log x = t

∴1/xdx=dt

$$\Rightarrow \int \frac{(1+\log x)^2}{x} dx = \int t^2 dt$$
$$= \frac{t^3}{3} + C$$
$$= \frac{(1+\log x)^3}{3} + C$$

Question 36. $\frac{(x+1)(x+\log x)^2}{x}$

Solution :

$$\frac{(x+1)(x+\log x)^2}{x}=\left(\frac{x+1}{x}\right)(x+\log x)^2=\left(1+\frac{1}{x}\right)(x+\log x)^2$$

Let $(x+\log x)=t$

$$\therefore \left(1+\frac{1}{x}\right)dx=dt$$

$$\Rightarrow \int\left(1+\frac{1}{x}\right)(x+\log x)^2\,dx=\int t^2\,dt$$
$$=\frac{t^3}{3}+C$$
$$=\frac{1}{3}(x+\log x)^3+C$$

Question 37. $\dfrac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}$

Solution :

Let $x^4 = t$

$\therefore 4x^3\,dx = dt$

$$\Rightarrow \int\frac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}dx=\frac{1}{4}\int\frac{\sin\left(\tan^{-1}t\right)}{1+t^2}dt \qquad \ldots(1)$$

Let $\tan^{-1}t=u$

$$\therefore \frac{1}{1+t^2}dt=du$$

From (1), we obtain

$$\int\frac{x^3\sin\left(\tan^{-1}x^4\right)dx}{1+x^8}=\frac{1}{4}\int\sin u\,du$$
$$=\frac{1}{4}(-\cos u)+C$$

$$=\frac{-1}{4}\cos\left(\tan^{-1}t\right)+C$$
$$=\frac{-1}{4}\cos\left(\tan^{-1}x^4\right)+C$$

Choose the correct answer in Exercise 38 and 39.

Question 38. $\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}$ dx **equals**

(A) $10^x - x^{10} + C$
(B) $10^x + x^{10} + C$
(C) $(10^x - x^{10})^{-1} + C$
(D) $\log(10^x + x^{10}) + C$

Solution :

Let $x^{10} + 10^x = t$

$\therefore \left(10x^9 + 10^x \log_e 10\right) dx = dt$

$\Rightarrow \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx = \int \frac{dt}{t}$

$= \log t + C$
$= \log\left(10^x + x^{10}\right) + C$

Therefore, option (D) is correct.

Question 39. $\int \frac{dx}{\sin^2 x \cos^2 x}$ **equals**

(A) $\tan x + \cot x + C$
(B) $\tan x - \cot x + C$
(C) $\tan x \cot x + C$
(D) $\tan x - \cot 2x + C$

Solution :

$$\text{Let } I = \int \frac{dx}{\sin^2 x \cos^2 x}$$
$$= \int \frac{1}{\sin^2 x \cos^2 x} dx$$
$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$$
$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$$
$$= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx$$
$$= \tan x - \cot x + C$$

Therefore, option (B) is correct.

Exercise 7.3 Page: 243

Find the integrals of the following functions in Exercises 1 to 9.

Question 1. $\sin^2(2x + 5)$

Solution :

$$\sin^2(2x+5) = \frac{1-\cos 2(2x+5)}{2} = \frac{1-\cos(4x+10)}{2}$$
$$\Rightarrow \int \sin^2(2x+5)\, dx = \int \frac{1-\cos(4x+10)}{2} dx$$
$$= \frac{1}{2}\int 1\, dx - \frac{1}{2}\int \cos(4x+10)\, dx$$
$$= \frac{1}{2}x - \frac{1}{2}\left(\frac{\sin(4x+10)}{4}\right) + C$$
$$= \frac{1}{2}x - \frac{1}{8}\sin(4x+10) + C$$

Question 2. sin 3x cos4x

Solution :

It is known that, $\sin A\cos B=\frac{1}{2}\{\sin(A+B)+\sin(A-B)\}$

$$\begin{aligned}\therefore \int \sin 3x\cos 4x\ dx &= \frac{1}{2}\int\{\sin(3x+4x)+\sin(3x-4x)\}\ dx\\ &=\frac{1}{2}\int\{\sin 7x+\sin(-x)\}\ dx\\ &=\frac{1}{2}\int\{\sin 7x-\sin x\}\ dx\\ &=\frac{1}{2}\int \sin 7x\ dx-\frac{1}{2}\int \sin x\ dx\\ &=\frac{1}{2}\left(\frac{-\cos 7x}{7}\right)-\frac{1}{2}(-\cos x)+C\\ &=\frac{-\cos 7x}{14}+\frac{\cos x}{2}+C\end{aligned}$$

Question 3. cos 2*x* cos 4*x* cos 6*x*

Solution **:**

It is known that, $\cos A\cos B=\frac{1}{2}\{\cos(A+B)+\cos(A-B)\}$

$$\begin{aligned}\therefore \int \cos 2x(\cos 4x\cos 6x)dx &= \int \cos 2x\left[\frac{1}{2}\{\cos(4x+6x)+\cos(4x-6x)\}\right] dx\\ &=\frac{1}{2}\int\{\cos 2x\cos 10x+\cos 2x\cos(-2x)\}\ dx\\ &=\frac{1}{2}\int\{\cos 2x\cos 10x+\cos^2 2x\}\ dx\\ &=\frac{1}{2}\int\left[\left\{\frac{1}{2}\cos(2x+10x)+\cos(2x-10x)\right\}+\left(\frac{1+\cos 4x}{2}\right)\right] dx\\ &=\frac{1}{4}\int(\cos 12x+\cos 8x+1+\cos 4x)\ dx\\ &=\frac{1}{4}\left[\frac{\sin 12x}{12}+\frac{\sin 8x}{8}+x+\frac{\sin 4x}{4}\right]+C\end{aligned}$$

Question 4. sin^3 (2x + 1)

Let $I = \int \sin^3(2x+1)$

$$\Rightarrow \int \sin^3(2x+1)\,dx = \int \sin^2(2x+1)\cdot\sin(2x+1)\,dx$$
$$= \int\left(1-\cos^2(2x+1)\right)\sin(2x+1)\,dx$$

Let $\cos(2x+1) = t$

$$\Rightarrow -2\sin(2x+1)\,dx = dt$$

$$\Rightarrow \sin(2x+1)\,dx = \frac{-dt}{2}$$

$$\Rightarrow I = \frac{-1}{2}\int\left(1-t^2\right)dt$$
$$= \frac{-1}{2}\left\{t-\frac{t^3}{3}\right\}$$
$$= \frac{-1}{2}\left\{\cos(2x+1)-\frac{\cos^3(2x+1)}{3}\right\}$$
$$= \frac{-\cos(2x+1)}{2}+\frac{\cos^3(2x+1)}{6}+C$$

Question 5. sin^3 x cos^3 x

$$\text{Let } I = \int \sin^3 x\cos^3 x\cdot dx$$
$$= \int \cos^3 x\cdot\sin^2 x\cdot\sin x\cdot dx$$
$$= \int \cos^3 x\left(1-\cos^2 x\right)\sin x\cdot dx$$

Let $\cos x = t$

$$\Rightarrow -\sin x\cdot dx = dt$$
$$\Rightarrow I = -\int t^3\left(1-t^2\right)dt$$
$$= -\int\left(t^3-t^5\right)dt$$
$$= -\left\{\frac{t^4}{4}-\frac{t^6}{6}\right\}+C$$
$$= -\left\{\frac{\cos^4 x}{4}-\frac{\cos^6 x}{6}\right\}+C$$
$$= \frac{\cos^6 x}{6}-\frac{\cos^4 x}{4}+C$$

Question 6. sin x sin 2x sin 3x

It is known that, $\sin A \sin B = \frac{1}{2}\{\cos(A-B)-\cos(A+B)\}$

$$\therefore \int \sin x \sin 2x \sin 3x \, dx = \int \left[\sin x \cdot \frac{1}{2}\{\cos(2x-3x)-\cos(2x+3x)\} \right] dx$$

$$= \frac{1}{2}\int \left(\sin x \cos(-x) - \sin x \cos 5x\right) dx$$

$$= \frac{1}{2}\int \left(\sin x \cos x - \sin x \cos 5x\right) dx$$

$$= \frac{1}{2}\int \frac{\sin 2x}{2} dx - \frac{1}{2}\int \sin x \cos 5x \, dx$$

$$= \frac{1}{4}\left[\frac{-\cos 2x}{2}\right] - \frac{1}{2}\int \left\{\frac{1}{2}\sin(x+5x)+\sin(x-5x)\right\} dx$$

$$= \frac{-\cos 2x}{8} - \frac{1}{4}\int \left(\sin 6x + \sin(-4x)\right) dx$$

$$= \frac{-\cos 2x}{8} - \frac{1}{4}\left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{4}\right] + C$$

$$= \frac{-\cos 2x}{8} - \frac{1}{8}\left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{2}\right] + C$$

$$= \frac{1}{8}\left[\frac{\cos 6x}{3} - \frac{\cos 4x}{2} - \cos 2x\right] + C$$

Question 7. sin 4*x* sin 8*x*

Solution: Itisknownthat,
sin A . sin B = 12cosA-B-cosA+B

∴∫sin4x sin8x dx=∫12cos4x-8x-cos4x+8xdx

=12∫cos-4x-cos12xdx

=12∫cos4x-cos12xdx

=12sin4x4-sin12x12+C

Question 8. $\dfrac{1-\cos x}{1+\cos x}$

$$\frac{1-\cos x}{1+\cos x}=\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} \qquad \left[2\sin^2\frac{x}{2}=1-\cos x \text{ and } 2\cos^2\frac{x}{2}=1+\cos x\right]$$

$$=\tan^2\frac{x}{2}$$

$$=\left(\sec^2\frac{x}{2}-1\right)$$

$$\therefore \int\frac{1-\cos x}{1+\cos x}dx=\int\left(\sec^2\frac{x}{2}-1\right)dx$$

$$=\left[\frac{\tan\frac{x}{2}}{\frac{1}{2}}-x\right]+C$$

$$=2\tan\frac{x}{2}-x+C$$

Question 9. $\dfrac{\cos x}{1+\cos x}$

Solution :

Find the integrals of the following functions in Exercises 10 to 18.

$$\frac{\cos x}{1+\cos x}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} \qquad \left[\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2} \text{ and } \cos x=2\cos^2\frac{x}{2}-1\right]$$

$$=\frac{1}{2}\left[1-\tan^2\frac{x}{2}\right]$$

$$\therefore \int\frac{\cos x}{1+\cos x}dx=\frac{1}{2}\int\left(1-\tan^2\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\int\left(1-\sec^2\frac{x}{2}+1\right)dx$$

$$=\frac{1}{2}\int\left(2-\sec^2\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\left[2x-\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right]+C$$

$$=x-\tan\frac{x}{2}+C$$

Question 10. $\sin^4 x$

Solution :

$$\sin^4 x = \sin^2 x \sin^2 x$$
$$= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)$$
$$= \frac{1}{4}(1-\cos 2x)^2$$
$$= \frac{1}{4}\left[1+\cos^2 2x - 2\cos 2x\right]$$
$$= \frac{1}{4}\left[1+\left(\frac{1+\cos 4x}{2}\right) - 2\cos 2x\right]$$
$$= \frac{1}{4}\left[1+\frac{1}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]$$
$$= \frac{1}{4}\left[\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]$$

$$\therefore \int \sin^4 x\, dx = \frac{1}{4}\int\left[\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]dx$$
$$= \frac{1}{4}\left[\frac{3}{2}x+\frac{1}{2}\left(\frac{\sin 4x}{4}\right) - \frac{2\sin 2x}{2}\right] + C$$
$$= \frac{1}{8}\left[3x+\frac{\sin 4x}{4} - 2\sin 2x\right] + C$$
$$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Question 11. $\cos^4 2x$

Solution :

$$\begin{aligned}\cos^4 2x &= \left(\cos^2 2x\right)^2 \\ &= \left(\frac{1+\cos 4x}{2}\right)^2 \\ &= \frac{1}{4}\left[1+\cos^2 4x + 2\cos 4x\right] \\ &= \frac{1}{4}\left[1+\left(\frac{1+\cos 8x}{2}\right)+2\cos 4x\right] \\ &= \frac{1}{4}\left[1+\frac{1}{2}+\frac{\cos 8x}{2}+2\cos 4x\right] \\ &= \frac{1}{4}\left[\frac{3}{2}+\frac{\cos 8x}{2}+2\cos 4x\right]\end{aligned}$$

$$\therefore \int \cos^4 2x\, dx = \int\left(\frac{3}{8}+\frac{\cos 8x}{8}+\frac{\cos 4x}{2}\right)dx$$

$$= \frac{3}{8}x + \frac{\sin 8x}{64} + \frac{\sin 4x}{8} + C$$

Question 12. $\dfrac{\sin^2 x}{1+\cos x}$

Solution :

Question13. $\dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$

$$\frac{\sin^2 x}{1+\cos x} = \frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}} \quad \left[\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2};\ \cos x = 2\cos^2\frac{x}{2}-1\right]$$

$$= \frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$= 2\sin^2\frac{x}{2}$$

$$= 1-\cos x$$

$$\therefore \int \frac{\sin^2 x}{1+\cos x}dx = \int (1-\cos x)dx$$

$$= x - \sin x + C$$

$$\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha} = \frac{-2\sin\frac{2x+2\alpha}{2}\sin\frac{2x-2\alpha}{2}}{-2\sin\frac{x+\alpha}{2}\sin\frac{x-\alpha}{2}} \qquad \left[\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$= \frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$= \frac{\left[2\sin\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x+\alpha}{2}\right)\right]\left[2\sin\left(\frac{x-\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)\right]}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$= 4\cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)$$

$$= 2\left[\cos\left(\frac{x+\alpha}{2}+\frac{x-\alpha}{2}\right)+\cos\frac{x+\alpha}{2}-\frac{x-\alpha}{2}\right]$$

$$= 2\left[\cos(x)+\cos\alpha\right]$$

$$= 2\cos x + 2\cos\alpha$$

$$\therefore \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}dx = \int 2\cos x + 2\cos\alpha$$

$$= 2\left[\sin x + x\cos\alpha\right] + C$$

Question 14. $\frac{\cos x - \sin x}{1+\sin 2x}$

Solution :

$$\frac{\cos x-\sin x}{1+\sin 2x}=\frac{\cos x-\sin x}{\left(\sin^2 x+\cos^2 x\right)+2\sin x\cos x}$$

$$\left[\sin^2 x+\cos^2 x=1;\ \ \sin 2x=2\sin x\cos x\right]$$

$$=\frac{\cos x-\sin x}{\left(\sin x+\cos x\right)^2}$$

Let $\sin x+\cos x=t$

$\therefore \left(\cos x-\sin x\right)dx=dt$

$$\Rightarrow \int\frac{\cos x-\sin x}{1+\sin 2x}dx=\int\frac{\cos x-\sin x}{\left(\sin x+\cos x\right)^2}dx$$

$$=\int\frac{dt}{t^2}$$

$$=\int t^{-2}dt$$

$$=-t^{-1}+C$$

$$=-\frac{1}{t}+C$$

$$=\frac{-1}{\sin x+\cos x}+C$$

Question 15. $\tan^3 2x \sec 2x$

Solution **:**

$$\tan^3 2x \sec 2x = \tan^2 2x \tan 2x \sec 2x$$
$$= \left(\sec^2 2x - 1\right) \tan 2x \sec 2x$$
$$= \sec^2 2x \cdot \tan 2x \sec 2x - \tan 2x \sec 2x$$
$$\therefore \int \tan^3 2x \sec 2x \, dx = \int \sec^2 2x \tan 2x \sec 2x \, dx - \int \tan 2x \sec 2x \, dx$$
$$= \int \sec^2 2x \tan 2x \sec 2x \, dx - \frac{\sec 2x}{2} + C$$

Let $\sec 2x = t$

$\therefore 2 \sec 2x \tan 2x \, dx = dt$

$$\therefore \int \tan^3 2x \sec 2x \, dx = \frac{1}{2} \int t^2 dt - \frac{\sec 2x}{2} + C$$
$$= \frac{t^3}{6} - \frac{\sec 2x}{2} + C$$
$$= \frac{(\sec 2x)^3}{6} - \frac{\sec 2x}{2} + C$$

Question 16. $\tan^4 x$

$$\begin{aligned}
&\tan^4 x \\
&= \tan^2 x \cdot \tan^2 x \\
&= \left(\sec^2 x - 1\right)\tan^2 x \\
&= \sec^2 x \tan^2 x - \tan^2 x \\
&= \sec^2 x \tan^2 x - \left(\sec^2 x - 1\right) \\
&= \sec^2 x \tan^2 x - \sec^2 x + 1
\end{aligned}$$

$$\therefore \int \tan^4 x\, dx = \int \sec^2 x \tan^2 x\, dx - \int \sec^2 x\, dx + \int 1 \cdot dx$$
$$= \int \sec^2 x \tan^2 x\, dx - \tan x + x + C \qquad \ldots(1)$$

Consider $\int \sec^2 x \tan^2 x\, dx$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow \int \sec^2 x \tan^2 x dx = \int t^2 dt = \frac{t^3}{3} = \frac{\tan^3 x}{3}$$

From equation (1), we obtain

$$\int \tan^4 x\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C$$

Solution :Question 17.

$$\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$$

Solution :

$$\begin{aligned}
\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} &= \frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x} \\
&= \tan x \sec x + \cot x \operatorname{cosec} x
\end{aligned}$$

$$\therefore \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}\, dx = \int \left(\tan x \sec x + \cot x \operatorname{cosec} x\right) dx$$
$$= \sec x - \operatorname{cosec} x + C$$

Question 18.

$$\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$$

Solution :

.

$$\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos 2x + (1 - \cos 2x)}{\cos^2 x} \qquad [\cos 2x = 1 - 2\sin^2 x]$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$$\therefore \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\, dx = \int \sec^2 x\, dx = \tan x + C$$

Question 19. $\dfrac{1}{\sin x \cos^3 x}$

Solution :

$$\frac{1}{\sin x \cos^3 x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x} = \frac{\sin x}{\cos^3 x} + \frac{1}{\sin x \cos x}$$

$$\Rightarrow \frac{1}{\sin x \cos^3 x} = \tan x \sec^2 x + \frac{\frac{1}{\cos^2 x}}{\frac{\sin x \cos x}{\cos^2 x}} = \tan x \sec^2 x + \frac{\sec^2 x}{\tan x}$$

$$\therefore \int \frac{1}{\sin x \cos^3 x} dx = \int \tan x \sec^2 x\, dx + \int \frac{\sec^2 x}{\tan x}\, dx$$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow \int \frac{1}{\sin x \cos^3 x} dx = \int t\,dt + \int \frac{1}{t}\, dt$$

$$= \frac{t^2}{2} + \log|t| + C$$

$$= \frac{1}{2}\tan^2 x + \log|\tan x| + C$$

Question20. $\dfrac{\cos 2x}{(\cos x + \sin x)^2}$

Solution :20

$$\frac{\cos 2x}{(\cos x+\sin x)^2}=\frac{\cos 2x}{\cos^2 x+\sin^2 x+2\sin x\cos x}=\frac{\cos 2x}{1+\sin 2x}$$

$$\therefore\ \int\frac{\cos 2x}{(\cos x+\sin x)^2}\,dx=\int\frac{\cos 2x}{(1+\sin 2x)}\,dx$$

Let $1+\sin 2x=t$

$\Rightarrow 2\cos 2x\ dx=dt$

$$\therefore\ \int\frac{\cos 2x}{(\cos x+\sin x)^2}\,dx=\frac{1}{2}\int\frac{1}{t}dt$$

$$=\frac{1}{2}\log|t|+\text{C}$$

$$=\frac{1}{2}\log|1+\sin 2x|+\text{C}$$

$$=\frac{1}{2}\log\left|(\sin x+\cos x)^2\right|+\text{C}$$

$$=\log|\sin x+\cos x|+\text{C}$$

Question 21. $\sin^{-1}(\cos x)$

Solution :

$\sin^{-1}(\cos x)$

Let $\cos x = t$

Then, $\sin x = \sqrt{1-t^2}$

$\Rightarrow (-\sin x)\,dx = dt$

$dx = \dfrac{-dt}{\sin x}$

$dx = \dfrac{-dt}{\sqrt{1-t^2}}$

$$\therefore \int \sin^{-1}(\cos x)\,dx = \int \sin^{-1} t \left(\frac{-dt}{\sqrt{1-t^2}} \right)$$

$$= -\int \frac{\sin^{-1} t}{\sqrt{1-t^2}} dt$$

Let $\sin^{-1} t = u$

$\Rightarrow \dfrac{1}{\sqrt{1-t^2}} dt = du$

$$\therefore \int \sin^{-1}(\cos x)\,dx = \int 4 du$$

$$= -\frac{u^2}{2} + C$$

$$= \frac{-\left(\sin^{1} t\right)^2}{2} + C$$

$$= \frac{-\left[\sin^{-1}(\cos x)\right]^2}{2} + C \qquad \ldots(1)$$

It is known that,

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\therefore \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x) = \left(\frac{\pi}{2} - x\right)$$

Substituting in equation (1), we obtain

$$\int \sin^{-1}(\cos x)\, dx = \frac{-\left[\frac{\pi}{2} - x\right]^2}{2} + C$$

$$= -\frac{1}{2}\left(\frac{\pi^2}{2} + x^2 - \pi x\right) + C$$

$$= -\frac{\pi^2}{8} - \frac{x^2}{2} + \frac{1}{2}\pi x + C$$

$$= \frac{\pi x}{2} - \frac{x^2}{2} + \left(C - \frac{\pi^2}{8}\right)$$

$$= \frac{\pi x}{2} - \frac{x^2}{2} + C_1$$

Question 22. $\frac{1}{\cos(x-a)\cos(x-b)}$

Solution : Choose the correct answer in Exercise 23 and 24.

$$\frac{1}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\frac{[\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)]}{\cos(x-a)\cos(x-b)}$$

$$= \frac{1}{\sin(a-b)}[\tan(x-b)-\tan(x-a)]$$

$$\Rightarrow \int\frac{1}{\cos(x-a)\cos(x-b)}dx = \frac{1}{\sin(a-b)}\int[\tan(x-b)-\tan(x-a)]dx$$

$$= \frac{1}{\sin(a-b)}\left[-\log|\cos(x-b)|+\log|\cos(x-a)|\right]$$

$$= \frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right]+C$$

Question 23.is equal to: $\int\frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx$ is equal to

A. tan *x* + cot *x* + C

B. tan *x* + cosec *x* + C

C. − tan *x* + cot *x* + C

D. tan *x* + sec *x* + C

Solution :

$$\int\frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx = \int\left(\frac{\sin^2 x}{\sin^2 x\cos^2 x}-\frac{\cos^2 x}{\sin^2 x\cos^2 x}\right)dx$$

$$= \int\left(\sec^2 x-\text{cosec}^2 x\right)dx$$

$$= \tan x+\cot x+C$$

Therefore, option (A) is correct.

Question 24. is equal to: $\int \frac{e^x(1+x)}{\cos^2(e^x x)}dx$

A. − cot ($e^x x$) + C

B. tan (xe^x) + C

C. tan (e^x) + C

D. cot (e^x) + C

Solution :

Let I = $\int \frac{e^x(1+x)}{\cos^2(e^x x)}dx$

Let $e^x x = t$

$$\Rightarrow (e^x \cdot x + e^x \cdot 1)dx = dt$$
$$e^x(x+1)dx = dt$$
$$\therefore \int \frac{e^x(1+x)}{\cos^2(e^x x)}dx = \int \frac{dt}{\cos^2 t}$$
$$= \int \sec^2 t\ dt$$
$$= \tan t + C$$
$$= \tan(e^x \cdot x) + C$$

Therefore, option (B) is correct.

Exercise 7.4 Page: 251

Integrate the following functions in Exercises 1 to 9.

Question 1. $\frac{3x^2}{x^6+1}$

Solution :

Let $x^3 = t$

$\therefore 3x^2\ dx = dt$

$$\Rightarrow \int \frac{3x^2}{x^6+1}dx = \int \frac{dt}{t^2+1}$$
$$= \tan^{1} t + C$$
$$= \tan^{-1}\left(x^3\right) + C$$

Question 2. $\frac{1}{\sqrt{1+4x^2}}$

Solution :

Let $2x = t$

$\therefore 2dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$$
$$= \frac{1}{2}\left[\log\left|t+\sqrt{t^2+1}\right|\right] + C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$
$$= \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right| + C$$

Question 3. $\frac{1}{\sqrt{(2-x)^2+1}}$

Solution :

Let $2 - x = t$

$\Rightarrow -dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{1}{\sqrt{t^2+1}}dt$$
$$= -\log\left|t+\sqrt{t^2+1}\right| + C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$
$$= -\log\left|2-x+\sqrt{(2-x)^2+1}\right| + C$$
$$= \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right| + C$$

Question 4. $\frac{1}{\sqrt{9-25x^2}}$

Solution :

Let $5x = t$

$\therefore 5dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{9-25x^2}}\,dx = \frac{1}{5}\int \frac{1}{9-t^2}\,dt$$
$$= \frac{1}{5}\int \frac{1}{\sqrt{3^2-t^2}}\,dt$$
$$= \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right) + C$$
$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

Question 5. $\frac{3x}{1+2x^4}$

Solution :

Let $\sqrt{2}x^2 = t$

$\therefore\ 2\sqrt{2}x\ dx = dt$

$$\Rightarrow \int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\int \frac{dt}{1+t^2}$$
$$= \frac{3}{2\sqrt{2}}\left[\tan^{-1} t\right] + C$$
$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$$

Question 6. $\frac{x^2}{1-x^6}$

Solution :

Let $x^3 = t$

$\therefore 3x^2\,dx = dt$

$$\Rightarrow \int \frac{x^2}{1-x^6}dx = \frac{1}{3}\int \frac{dt}{1-t^2}$$
$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$
$$= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

Question 7. $\frac{x-1}{\sqrt{x^2-1}}$

Solution :

$$\int \frac{x-1}{\sqrt{x^2-1}}\,dx = \int \frac{x}{\sqrt{x^2-1}}\,dx - \int \frac{1}{\sqrt{x^2-1}}\,dx \quad \ldots(1)$$

For $\int \frac{x}{\sqrt{x^2-1}}dx$, let $x^2 - 1 = t \Rightarrow 2x\,dx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$
$$= \frac{1}{2}\int t^{-\frac{1}{2}}dt$$
$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right]$$
$$= \sqrt{t}$$
$$= \sqrt{x^2-1}$$

From (1), we obtain

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx \qquad \left[\int \frac{1}{\sqrt{x^2-a^2}}dt = \log\left|x+\sqrt{x^2-a^2}\right|\right]$$
$$= \sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right| + C$$

Question 8. $\frac{x^2}{\sqrt{x^6+a^6}}$

Solution :

Let $x^3 = t$

$\therefore 3x^2\, dx = dt$

$$\therefore \int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+\left(a^3\right)^2}}$$
$$= \frac{1}{3}\log\left|t+\sqrt{t^2+a^6}\right| + C$$
$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right| + C$$

Question 9. $\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

Solution :

Let $\tan x = t$

$\therefore \sec^2 x\, dx = dt$

$$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx = \int \frac{dt}{\sqrt{t^2+2^2}}$$
$$= \log\left|t+\sqrt{t^2+4}\right| + C$$
$$= \log\left|\tan x+\sqrt{\tan^2 x+4}\right| + C$$

Integrate the following functions in Exercises 10 to 18.

Question 10. $\frac{1}{\sqrt{x^2+2x+2}}$

Solution :

$$\int \frac{1}{\sqrt{x^2+2x+2}}\,dx = \int \frac{1}{\sqrt{(x+1)^2+(1)^2}}\,dx$$

Let $x+1=t$

$\therefore\ dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+2x+2}}\,dx = \int \frac{1}{\sqrt{t^2+1}}\,dt$$

$$= \log\left|t+\sqrt{t^2+1}\right| + C$$

$$= \log\left|(x+1)+\sqrt{(x+1)^2+1}\right| + C$$

$$= \log\left|(x+1)+\sqrt{x^2+2x+2}\right| + C$$

Question 11. $\frac{1}{9x^2+6x+5}$

Solution :

$$\int \frac{1}{9x^2+6x+5}\,dx = \int \frac{1}{(3x+1)^2+2^2}\,dx$$

Let $(3x+1)=t$

$\therefore 3\,dx = dt$

$$\Rightarrow \int \frac{1}{(3x+1)^2+2^2}\,dx = \frac{1}{3}\int \frac{1}{t^2+2^2}\,dt$$

$$= \frac{1}{3\times 2}\tan^{-1}\frac{t}{2} + C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right) + C$$

Question 12 . $\frac{1}{\sqrt{7-6x-x^2}}$

Solution :

$7-6x-x^2$ can be written as $7-\left(x^2+6x+9-9\right)$.

Therefore,

$$7-\left(x^2+6x+9-9\right)$$

$$=16-\left(x^2+6x+9\right)$$

$$=16-(x+3)^2$$

$$=(4)^2-(x+3)^2$$

$$\therefore \int \frac{1}{\sqrt{7-6x-x^2}}dx = \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx$$

Let $x+3=t$

$\Rightarrow dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx = \int \frac{1}{\sqrt{(4)^2-(t)^2}}dt$$

$$=\sin^{-1}\left(\frac{t}{4}\right)+C$$

$$=\sin^{-1}\left(\frac{x+3}{4}\right)+C$$

Question 13. $\frac{1}{\sqrt{(x-1)(x-2)}}$

Solution :

$(x-1)(x-2)$ can be written as x^2-3x+2.

Therefore,

x^2-3x+2

$=x^2-3x+\frac{9}{4}-\frac{9}{4}+2$

$=\left(x-\frac{3}{2}\right)^2-\frac{1}{4}$

$=\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$

$$\therefore \int \frac{1}{\sqrt{(x-1)(x-2)}}dx=\int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

Let $x-\frac{3}{2}=t$

$\therefore\ dx=dt$

$$\Rightarrow \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx=\int \frac{1}{\sqrt{t^2-\left(\frac{1}{2}\right)^2}}dt$$

$$=\log\left|t+\sqrt{t^2-\left(\frac{1}{2}\right)^2}\right|+C$$

$$=\log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C\ \frac{1}{\sqrt{8+3x-x^2}}$$

Question 14.

Solution :

$8+3x-x^2$ can be written as $8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$.

Therefore,

$$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

$$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

$$\Rightarrow \int \frac{1}{\sqrt{8+3x-x^2}}dx=\int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx$$

Let $x-\frac{3}{2}=t$

$\therefore\ dx=dt$

$$\Rightarrow \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx=\int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-t^2}}dt$$

$$=\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$$

Question 15. $\frac{1}{\sqrt{(x-a)(x-b)}}$

Solution :

$(x-a)(x-b)$ can be written as $x^2-(a+b)x+ab$.

Therefore,

$$x^2-(a+b)x+ab$$

$$=x^2-(a+b)x+\frac{(a+b)^2}{4}-\frac{(a+b)^2}{4}+ab$$

$$=\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}$$

$$\Rightarrow \int \frac{1}{\sqrt{(x-a)(x-b)}}dx=\int \frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx$$

Let $x-\left(\frac{a+b}{2}\right)=t$

$\therefore\ dx=dt$

$$\Rightarrow \int \frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx=\int \frac{1}{\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}}dt$$

$$=\log\left|t+\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}\right|+C$$

$$=\log\left|\left\{x-\left(\frac{a+b}{2}\right)\right\}+\sqrt{(x-a)(x-b)}\right|+C$$

Question 16. $\frac{4x+1}{\sqrt{2x^2+x-3}}$

Solution **:**

$$\text{Let } 4x+1 = A\frac{d}{dx}\left(2x^2+x-3\right)+B$$
$$\Rightarrow 4x+1 = A(4x+1)+B$$
$$\Rightarrow 4x+1 = 4Ax+A+B$$

Equating the coefficients of x and constant term on both sides, we obtain

$4A = 4 \Rightarrow A = 1$

$A + B = 1 \Rightarrow B = 0$

Let $2x^2 + x - 3 = t$

$\therefore (4x + 1)\, dx = dt$

$$\Rightarrow \int \frac{4x+1}{\sqrt{2x^2+x-3}}dx = \int \frac{1}{\sqrt{t}}dt$$
$$= 2\sqrt{t} + \text{C}$$
$$= 2\sqrt{2x^2+x-3} + \text{C}$$

Question 17. $\dfrac{x+2}{\sqrt{x^2-1}}$

Solution :

$$\text{Let } x+2 = A\frac{d}{dx}\left(x^2-1\right)+B \qquad ...(1)$$

$$\Rightarrow x+2 = A(2x)+B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$B = 2$$

From (1), we obtain

$$(x+2) = \frac{1}{2}(2x)+2$$

$$\text{Then, } \int \frac{x+2}{\sqrt{x^2-1}}dx = \int \frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$$

$$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx + \int \frac{2}{\sqrt{x^2-1}}dx \qquad ...(2)$$

$$\text{In } \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx, \text{ let } x^2-1 = t \Rightarrow 2x\,dx = dt$$

$$\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2}\left[2\sqrt{t}\right]$$

$$= \sqrt{t}$$

$$= \sqrt{x^2-1}$$

$$\text{Then, } \int \frac{2}{\sqrt{x^2-1}}dx = 2\int \frac{1}{\sqrt{x^2-1}}dx = 2\log\left|x+\sqrt{x^2-1}\right|$$

From equation (2), we obtain

$$\int \frac{x+2}{\sqrt{x^2-1}}dx = \sqrt{x^2-1} + 2\log\left|x+\sqrt{x^2-1}\right| + C$$

Question18. $\frac{5x-2}{1+2x+3x^2}$

Solution :

$$\text{Let } 5x-2 = A\frac{d}{dx}\left(1+2x+3x^2\right)+B$$
$$\Rightarrow 5x-2 = A\left(2+6x\right)+B$$

Equating the coefficient of *x* and constant term on both sides, we obtain

$$5 = 6A \Rightarrow A = \frac{5}{6}$$
$$2A+B = -2 \Rightarrow B = -\frac{11}{3}$$
$$\therefore 5x-2 = \frac{5}{6}\left(2+6x\right)+\left(-\frac{11}{3}\right)$$
$$\Rightarrow \int\frac{5x-2}{1+2x+3x^2}\,dx = \int\frac{\frac{5}{6}\left(2+6x\right)-\frac{11}{3}}{1+2x+3x^2}dx$$
$$= \frac{5}{6}\int\frac{2+6x}{1+2x+3x^2}dx - \frac{11}{3}\int\frac{1}{1+2x+3x^2}dx$$
$$\text{Let } I_1 = \int\frac{2+6x}{1+2x+3x^2}dx \text{ and } I_2 = \int\frac{1}{1+2x+3x^2}dx$$
$$\therefore \int\frac{5x-2}{1+2x+3x^2}\,dx = \frac{5}{6}I_1 - \frac{11}{3}I_2 \qquad ...(1)$$

$$I_1 = \int\frac{2+6x}{1+2x+3x^2}dx$$
$$\text{Let } 1+2x+3x^2 = t$$
$$\Rightarrow \left(2+6x\right)dx = dt$$
$$\therefore I_1 = \int\frac{dt}{t}$$
$$I_1 = \log\left|t\right|$$
$$I_1 = \log\left|1+2x+3x^2\right| \qquad ...(2)$$

$$I_2 = \int \frac{1}{1+2x+3x^2}dx$$

$1+2x+3x^2$ can be written as $1+3\left(x^2+\frac{2}{3}x\right)$.

Therefore,

$$1+3\left(x^2+\frac{2}{3}x\right)$$

$$=1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)$$

$$=1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3}$$

$$=\frac{2}{3}+3\left(x+\frac{1}{3}\right)^2$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]$$

$$I_2=\frac{1}{3}\int\frac{1}{\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]}dx$$

$$=\frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$$

$$=\frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) \qquad ...(3)$$

Substituting equations (2) and (3) in equation (1), we obtain

$$\int\frac{5x-2}{1+2x+3x^2}dx=\frac{5}{6}\left[\log\left|1+2x+3x^2\right|\right]-\frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]+C$$

$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$$

Integrate the following functions in Exercises 19 to 23.

Question 19. $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$

Solution :

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}} = \frac{6x+7}{\sqrt{x^2-9x+20}}$$

$$\text{Let } 6x+7 = A\frac{d}{dx}\left(x^2-9x+20\right)+B$$
$$\Rightarrow 6x+7 = A(2x-9)+B$$

Equating the coefficients of x and constant term, we obtain

$2A = 6 \Rightarrow A = 3$

$-9A + B = 7 \Rightarrow B = 34$

$\therefore 6x + 7 = 3(2x - 9) + 34$

$$\int\frac{6x+7}{\sqrt{x^2-9x+20}} = \int\frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$
$$= 3\int\frac{2x-9}{\sqrt{x^2-9x+20}}dx + 34\int\frac{1}{\sqrt{x^2-9x+20}}dx$$

$$\text{Let } I_1 = \int\frac{2x-9}{\sqrt{x^2-9x+20}}dx \text{ and } I_2 = \int\frac{1}{\sqrt{x^2-9x+20}}dx$$

$$\therefore \int\frac{6x+7}{\sqrt{x^2-9x+20}} = 3I_1 + 34I_2 \quad \ldots(1)$$

Then,

$$I_1 = \int\frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

Let $x^2-9x+20 = t$

$\Rightarrow (2x-9)dx = dt$

$$\Rightarrow I_1 = \frac{dt}{\sqrt{t}}$$

$$I_1 = 2\sqrt{t}$$

$$I_1 = 2\sqrt{x^2 - 9x + 20} \qquad ...(2)$$

and $I_2 = \int \frac{1}{\sqrt{x^2 - 9x + 20}} dx$

$x^2 - 9x + 20$ can be written as $x^2 - 9x + 20 + \frac{81}{4} - \frac{81}{4}$.

Therefore,

$$x^2 - 9x + 20 + \frac{81}{4} - \frac{81}{4}$$

$$= \left(x - \frac{9}{2}\right)^2 - \frac{1}{4}$$

$$= \left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$\Rightarrow I_2 = \int \frac{1}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

$$I_2 = \log\left|\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right| \qquad ...(3)$$

Substituting equations (2) and (3) in (1), we obtain

$$\int \frac{6x+7}{\sqrt{x^2 - 9x + 20}} dx = 3\left[2\sqrt{x^2 - 9x + 20}\right] + 34 \log\left[\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right] + C$$

$$= 6\sqrt{x^2 - 9x + 20} + 34 \log\left[\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right] + C$$

Question 20. $\frac{x+2}{\sqrt{4x - x^2}}$

Solution :

$$\text{Let } x+2 = A\frac{d}{dx}\left(4x-x^2\right)+B$$

$$\Rightarrow x+2 = A\left(4-2x\right)+B$$

Equating the coefficients of *x* and constant term on both sides, we obtain

$$-2A = 1 \Rightarrow A = -\frac{1}{2}$$

$$4A+B = 2 \Rightarrow B = 4$$

$$\Rightarrow (x+2) = -\frac{1}{2}(4-2x)+4$$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = \int \frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}}dx$$

$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx + 4\int \frac{1}{\sqrt{4x-x^2}}dx$$

$$\text{Let } I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx \text{ and } I_2 \int \frac{1}{\sqrt{4x-x^2}}dx$$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}I_1 + 4I_2 \qquad ...(1)$$

$$\text{Then, } I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$$

$$\text{Let } 4x-x^2 = t$$

$$\Rightarrow (4-2x)dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{4x-x^2} \qquad ...(2)$$

$$I_2 = \int \frac{1}{\sqrt{4x - x^2}} dx$$

$$\Rightarrow 4x - x^2 = -\left(-4x + x^2\right)$$

$$= \left(-4x + x^2 + 4 - 4\right)$$

$$= 4 - \left(x - 2\right)^2$$

$$= \left(2\right)^2 - \left(x - 2\right)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{\left(2\right)^2 - \left(x - 2\right)^2}} dx = \sin^{-1}\left(\frac{x - 2}{2}\right) \qquad \ldots(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x + 2}{\sqrt{4x - x^2}} dx = -\frac{1}{2}\left(2\sqrt{4x - x^2}\right) + 4\sin^{-1}\left(\frac{x - 2}{2}\right) + C$$

$$= -\sqrt{4x - x^2} + 4\sin^{-1}\left(\frac{x - 2}{2}\right) + C$$

Question 21. $\frac{x + 2}{\sqrt{x^2 + 2x + 3}}$

Solution **:**

$$\int \frac{(x+2)}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}}dx$$
$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}}dx$$
$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}}dx$$
$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

Let $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}I_1 + I_2 \qquad ...(1)$$

Then, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$

Let $x^2 + 2x + 3 = t$

$\Rightarrow (2x + 2)\, dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3} \qquad ...(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$\Rightarrow x^2+2x+3 = x^2+2x+1+2 = (x+1)^2 + \left(\sqrt{2}\right)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}}dx = \log\left|(x+1)+\sqrt{x^2+2x+3}\right| \qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\left[2\sqrt{x^2+2x+3}\right] + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$
$$= \sqrt{x^2+2x+3} + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

Question 22. $\frac{x+3}{x^2-2x-5}$

Solution :

$$\text{Let } (x+3) = A\frac{d}{dx}(x^2-2x-5)+B$$
$$(x+3) = A(2x-2)+B$$

Equating the coefficients of *x* and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$
$$-2A+B = 3 \Rightarrow B = 4$$
$$\therefore (x+3) = \frac{1}{2}(2x-2)+4$$
$$\Rightarrow \int \frac{x+3}{x^2-2x-5}dx = \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}dx$$
$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx + 4\int \frac{1}{x^2-2x-5}dx$$

$$\text{Let } I_1 = \int \frac{2x-2}{x^2-2x-5}dx \text{ and } I_2 = \int \frac{1}{x^2-2x-5}dx$$
$$\therefore \int \frac{x+3}{(x^2-2x-5)}dx = \frac{1}{2}I_1 + 4I_2 \qquad ...(1)$$
$$\text{Then, } I_1 = \int \frac{2x-2}{x^2-2x-5}dx$$

$$\text{Let } x^2-2x-5 = t$$
$$\Rightarrow (2x-2)dx = dt$$
$$\Rightarrow I_1 = \int \frac{dt}{t} = \log|t| = \log|x^2-2x-5| \qquad ...(2)$$

$$I_2 = \int \frac{1}{x^2 - 2x - 5} dx$$

$$= \int \frac{1}{\left(x^2 - 2x + 1\right) - 6} dx$$

$$= \int \frac{1}{(x-1)^2 + \left(\sqrt{6}\right)^2} dx$$

$$= \frac{1}{2\sqrt{6}} \log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) \quad ...(3)$$

Substituting (2) and (3) in (1), we obtain

$$\int \frac{x+3}{x^2-2x-5} dx = \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{4}{2\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

$$= \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

Question 23. $\frac{5x+3}{\sqrt{x^2+4x+10}}$

Solution :

$$\text{Let } 5x+3 = A\frac{d}{dx}\left(x^2+4x+10\right)+B$$
$$\Rightarrow 5x+3 = A\left(2x+4\right)+B$$

Equating the coefficients of *x* and constant term, we obtain

$$2A = 5 \Rightarrow A = \frac{5}{2}$$
$$4A+B = 3 \Rightarrow B = -7$$
$$\therefore 5x+3 = \frac{5}{2}(2x+4)-7$$
$$\Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$$
$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{1}{\sqrt{x^2+4x+10}}dx$$
$$\text{Let } I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx \text{ and } I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$$
$$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}I_1 - 7I_2 \quad ...(1)$$
$$\text{Then, } I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$$

$$\text{Let } x^2+4x+10 = t$$
$$\therefore (2x+4)dx = dt$$
$$\Rightarrow I_1 = \int \frac{dt}{t} = 2\sqrt{t} = 2\sqrt{x^2+4x+10} \quad ...(2)$$
$$I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$$
$$= \int \frac{1}{\sqrt{\left(x^2+4x+4\right)+6}}dx$$
$$= \int \frac{1}{(x+2)^2+\left(\sqrt{6}\right)^2}dx$$
$$= \log\left|(x+2)\sqrt{x^2+4x+10}\right| \quad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}\left[2\sqrt{x^2+4x+10}\right] - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$
$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

Choose the correct answer in Exercise 24 and 25.

Question 24. equals $\int \frac{dx}{x^2+2x+2}$

A. $x\tan^{-1}(x+1)+C$

B. $\tan^{-1}(x+1)+C$

C. $(x+1)\tan^{-1}x+C$

D. $\tan^{-1}x+C$

Solution :

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$$
$$= \int \frac{1}{(x+1)^2+(1)^2}dx$$
$$= \left[\tan^{-1}(x+1)\right]+C$$

Therefore, option (B) is correct.

Question 25. equals $\int \frac{dx}{\sqrt{9x-4x^2}}$

(A) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

(B) $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

(C) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

(D) $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

Solution :

Therefore,option(B)iscorrect.

$$\int \frac{dx}{\sqrt{9x-4x^2}}$$

$$= \int \frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x\right)}}dx$$

$$= \int \frac{1}{-4\left(x^2-\frac{9}{4}x+\frac{81}{64}-\frac{81}{64}\right)}dx$$

$$= \int \frac{1}{\sqrt{-4\left[\left(x-\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]}}dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}}dx$$

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right)\right]+C \qquad \left(\int \frac{dy}{\sqrt{a^2-y^2}} = \sin^{-1}\frac{y}{a}+C\right)$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

Exercise 7.5 Page: 258

Question 1. $\frac{x}{(x+1)(x+2)}$

Solution :

Let $\frac{x}{(x+1)(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$

$\Rightarrow x=A(x+2)+B(x+1)$

Equating the coefficients of *x* and constant term, we obtain

$A + B = 1$

$2A + B = 0$

On solving, we obtain

$A = -1$ and $B = 2$

$$\therefore \frac{x}{(x+1)(x+2)}=\frac{-1}{(x+1)}+\frac{2}{(x+2)}$$

$$\Rightarrow \int\frac{x}{(x+1)(x+2)}dx=\int\frac{-1}{(x+1)}+\frac{2}{(x+2)}dx$$

$$=-\log|x+1|+2\log|x+2|+C$$

$$=\log(x+2)^2-\log|x+1|+C$$

$$=\log\frac{(x+2)^2}{(x+1)}+C$$

Question 2. $\frac{1}{x^2-9}$

Solution :

Let $\frac{1}{(x+3)(x-3)}=\frac{A}{(x+3)}+\frac{B}{(x-3)}$

$1=A(x-3)+B(x+3)$

Equating the coefficients of *x* and constant term, we obtain

A + *B* = 0

−3*A* + 3*B* = 1

On solving, we obtain

$A=-\frac{1}{6}$ and $B=\frac{1}{6}$

$$\therefore \frac{1}{(x+3)(x-3)}=\frac{-1}{6(x+3)}+\frac{1}{6(x-3)}$$

$$\Rightarrow \int\frac{1}{(x^2-9)}dx=\int\left(\frac{-1}{6(x+3)}+\frac{1}{6(x-3)}\right)dx$$

$$=-\frac{1}{6}\log|x+3|+\frac{1}{6}\log|x-3|+C$$

$$=\frac{1}{6}\log\left|\frac{(x-3)}{(x+3)}\right|+C$$

Question 3. $\frac{3x-1}{(x-1)(x-2)(x-3)}$

Solution :

Let $\frac{3x-1}{(x-1)(x-2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$

$3x-1=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$ ...(1)

Substituting *x* = 1, 2, and 3 respectively in equation (1), we obtain

A = 1, *B* = −5, and *C* = 4

$$\therefore \frac{3x-1}{(x-1)(x-2)(x-3)}=\frac{1}{(x-1)}-\frac{5}{(x-2)}+\frac{4}{(x-3)}$$

$$\Rightarrow \int\frac{3x-1}{(x-1)(x-2)(x-3)}dx=\int\left\{\frac{1}{(x-1)}-\frac{5}{(x-2)}+\frac{4}{(x-3)}\right\}dx$$

$$=\log|x-1|-5\log|x-2|+4\log|x-3|+C$$

uestion 4. $\frac{x}{(x-1)(x-2)(x-3)}$

Solution :

Let $\frac{x}{(x-1)(x-2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$

$x=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$ ...(1)

Substituting *x* = 1, 2, and 3 respectively in equation (1), we obtain $A=\frac{1}{2},\ B=-2,\ \text{and } C=\frac{3}{2}$

$$\therefore \frac{x}{(x-1)(x-2)(x-3)}=\frac{1}{2(x-1)}-\frac{2}{(x-2)}+\frac{3}{2(x-3)}$$

$$\Rightarrow \int\frac{x}{(x-1)(x-2)(x-3)}dx=\int\left\{\frac{1}{2(x-1)}-\frac{2}{(x-2)}+\frac{3}{2(x-3)}\right\}dx$$

$$=\frac{1}{2}\log|x-1|-2\log|x-2|+\frac{3}{2}\log|x-3|+C$$

Question 5. $\frac{2x}{x^2+3x+2}$

Solution :

Let $\frac{2x}{x^2+3x+2}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$

$$2x=A(x+2)+B(x+1) \qquad \ldots(1)$$

Substituting x = −1 and −2 in equation (1), we obtain

A = −2 and B = 4

$$\therefore \frac{2x}{(x+1)(x+2)}=\frac{-2}{(x+1)}+\frac{4}{(x+2)}$$

$$\Rightarrow \int\frac{2x}{(x+1)(x+2)}dx=\int\left\{\frac{4}{(x+2)}-\frac{2}{(x+1)}\right\}dx$$

$$=4\log|x+2|-2\log|x+1|+C$$

Question 6. $\frac{1-x^2}{x(1-2x)}$

Solution :

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(1-x^2)$ by $x(1-2x)$, we obtain

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left(\frac{2-x}{x(1-2x)}\right)$$

Let $\frac{2-x}{x(1-2x)} = \frac{A}{x} + \frac{B}{(1-2x)}$

$$\Rightarrow (2-x) = A(1-2x) + Bx \qquad ...(1)$$

Substituting x = 0 and $\frac{1}{2}$ in equation (1), we obtain

A = 2 and B = 3

$$\therefore \frac{2-x}{x(1-2x)} = \frac{2}{x} + \frac{3}{1-2x}$$

Substituting in equation (1), we obtain

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left\{\frac{2}{x} + \frac{3}{(1-2x)}\right\}$$

$$\Rightarrow \int \frac{1-x^2}{x(1-2x)}dx = \int\left\{\frac{1}{2} + \frac{1}{2}\left(\frac{2}{x} + \frac{3}{1-2x}\right)\right\}dx$$

$$= \frac{x}{2} + \log|x| + \frac{3}{2(-2)}\log|1-2x| + C$$

$$= \frac{x}{2} + \log|x| - \frac{3}{4}\log|1-2x| + C$$

Question7. $\frac{x}{(x^2+1)(x-1)}$

Solution :

Let $\frac{x}{(x^2+1)(x-1)}=\frac{Ax+B}{(x^2+1)}+\frac{C}{(x-1)}$

$$x=(Ax+B)(x-1)+C(x^2+1)$$
$$x=Ax^2-Ax+Bx-B+Cx^2+C$$

Equating the coefficients of x^2, x, and constant term, we obtain

$A + C = 0$

$-A + B = 1$

$-B + C = 0$

On solving these equations, we obtain

$$A=-\frac{1}{2}, B=\frac{1}{2}, \text{ and } C=\frac{1}{2}$$

From equation (1), we obtain

$$\therefore \frac{x}{(x^2+1)(x-1)}=\frac{\left(-\frac{1}{2}x+\frac{1}{2}\right)}{x^2+1}+\frac{\frac{1}{2}}{(x-1)}$$
$$\Rightarrow \int\frac{x}{(x^2+1)(x-1)}=-\frac{1}{2}\int\frac{x}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x-1}dx$$
$$=-\frac{1}{4}\int\frac{2x}{x^2+1}dx+\frac{1}{2}\tan^{-1}x+\frac{1}{2}\log|x-1|+C$$

Consider $\int\frac{2x}{x^2+1}dx$, let $(x^2+1)=t \Rightarrow 2x\,dx=dt$

$$\Rightarrow \int\frac{2x}{x^2+1}dx=\int\frac{dt}{t}=\log|t|=\log|x^2+1|$$

$$\therefore \int\frac{x}{(x^2+1)(x-1)}=-\frac{1}{4}\log|x^2+1|+\frac{1}{2}\tan^{-1}x+\frac{1}{2}\log|x-1|+C$$
$$=\frac{1}{2}\log|x-1|-\frac{1}{4}\log|x^2+1|+\frac{1}{2}\tan^{-1}x+C$$

Question8. $\frac{x}{(x-1)^2(x+2)}$

Solution :

Let $\frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$

$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$

Substituting $x = 1$, we obtain

$B = \frac{1}{3}$

Equating the coefficients of x^2 and constant term, we obtain

$A + C = 0$

$-2A + 2B + C = 0$

On solving, we obtain

$A = \frac{2}{9}$ and $C = \frac{-2}{9}$

$$\therefore \frac{x}{(x-1)^2(x+2)} = \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)}$$

$$\Rightarrow \int \frac{x}{(x-1)^2(x+2)} dx = \frac{2}{9}\int \frac{1}{(x-1)} dx + \frac{1}{3}\int \frac{1}{(x-1)^2} dx - \frac{2}{9}\int \frac{1}{(x+2)} dx$$

$$= \frac{2}{9}\log|x-1| + \frac{1}{3}\left(\frac{-1}{x-1}\right) - \frac{2}{9}\log|x+2| + C$$

$$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C$$

Question9. $\frac{3x+5}{x^3 - x^2 - x + 1}$

Solution :

$$\frac{3x+5}{x^3-x^2-x+1}=\frac{3x+5}{(x-1)^2(x+1)}$$

Let $\frac{3x+5}{(x-1)^2(x+1)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)}$

$$3x+5=A(x-1)(x+1)+B(x+1)+C(x-1)^2$$
$$3x+5=A(x^2-1)+B(x+1)+C(x^2+1-2x) \qquad ...(1)$$

Substituting x = 1 in equation (1), we obtain

B = 4

Equating the coefficients of x^2 and x, we obtain

$A + C = 0$

$B - 2C = 3$

On solving, we obtain

$A=-\frac{1}{2}$ and $C=\frac{1}{2}$

$$\therefore \frac{3x+5}{(x-1)^2(x+1)}=\frac{-1}{2(x-1)}+\frac{4}{(x-1)^2}+\frac{1}{2(x+1)}$$
$$\Rightarrow \int\frac{3x+5}{(x-1)^2(x+1)}dx=-\frac{1}{2}\int\frac{1}{x-1}dx+4\int\frac{1}{(x-1)^2}dx+\frac{1}{2}\int\frac{1}{(x+1)}dx$$
$$=-\frac{1}{2}\log|x-1|+4\left(\frac{-1}{x-1}\right)+\frac{1}{2}\log|x+1|+C$$
$$=\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|-\frac{4}{(x-1)}+C$$

Question10. $\frac{2x-3}{(x^2-1)(2x+3)}$

Solution :

$$\frac{2x-3}{(x^2-1)(2x+3)}=\frac{2x-3}{(x+1)(x-1)(2x+3)}$$

Let $\frac{2x-3}{(x+1)(x-1)(2x+3)}=\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{C}{(2x+3)}$

$$\Rightarrow (2x-3)=A(x-1)(2x+3)+B(x+1)(2x+3)+C(x+1)(x-1)$$
$$\Rightarrow (2x-3)=A(2x^2+x-3)+B(2x^2+5x+3)+C(x^2-1)$$
$$\Rightarrow (2x-3)=(2A+2B+C)x^2+(A+5B)x+(-3A+3B-C)$$

Equating the coefficients of x^2 and x, we obtain

$B=-\frac{1}{10}$, $A=\frac{5}{2}$, and $C=-\frac{24}{5}$

$$\therefore \frac{2x-3}{(x+1)(x-1)(2x+3)}=\frac{5}{2(x+1)}-\frac{1}{10(x-1)}-\frac{24}{5(2x+3)}$$
$$\Rightarrow \int\frac{2x-3}{(x^2-1)(2x+3)}dx=\frac{5}{2}\int\frac{1}{(x+1)}dx-\frac{1}{10}\int\frac{1}{x-1}dx-\frac{24}{5}\int\frac{1}{(2x+3)}dx$$
$$=\frac{5}{2}\log|x+1|-\frac{1}{10}\log|x-1|-\frac{24}{5\times 2}\log|2x+3|$$
$$=\frac{5}{2}\log|x+1|-\frac{1}{10}\log|x-1|-\frac{12}{5}\log|2x+3|+C$$

Question11. $\frac{5x}{(x+1)(x^2-4)}$

Solution :

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$$

Let $\frac{5x}{(x+1)(x+2)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-2)}$

$$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2) \qquad ...(1)$$

Substituting $x = -1, -2$, and 2 respectively in equation (1), we obtain

$$A = \frac{5}{3},\ B = -\frac{5}{2}, \text{ and } C = \frac{5}{6}$$

$$\therefore \frac{5x}{(x+1)(x+2)(x-2)} = \frac{5}{3(x+1)} - \frac{5}{2(x+2)} + \frac{5}{6(x-2)}$$

$$\Rightarrow \int \frac{5x}{(x+1)(x^2-4)} dx = \frac{5}{3}\int \frac{1}{(x+1)} dx - \frac{5}{2}\int \frac{1}{(x+2)} dx + \frac{5}{6}\int \frac{1}{(x-2)} dx$$

$$= \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x+2| + \frac{5}{6}\log|x-2| + C$$

Question12. $\frac{x^3+x+1}{x^2-1}$

Solution :

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(x^3 + x + 1)$ by $x^2 - 1$, we obtain

Integrate the following function in Exercises 13 to 17.

$$\frac{x^3+x+1}{x^2-1}=x+\frac{2x+1}{x^2-1}$$

Let $\frac{2x+1}{x^2-1}=\frac{A}{(x+1)}+\frac{B}{(x-1)}$

$$2x+1=A(x-1)+B(x+1) \qquad ...(1)$$

Substituting x = 1 and −1 in equation (1), we obtain

$$A=\frac{1}{2} \text{ and } B=\frac{3}{2}$$

$$\therefore \frac{x^3+x+1}{x^2-1}=x+\frac{1}{2(x+1)}+\frac{3}{2(x-1)}$$

$$\Rightarrow \int\frac{x^3+x+1}{x^2-1}dx=\int x\,dx+\frac{1}{2}\int\frac{1}{(x+1)}dx+\frac{3}{2}\int\frac{1}{(x-1)}dx$$

$$=\frac{x^2}{2}+\frac{1}{2}\log|x+1|+\frac{3}{2}\log|x-1|+C$$

Question13. $\frac{2}{(1-x)(1+x^2)}$

Solution :

Let $\frac{2}{(1-x)(1+x^2)} = \frac{A}{(1-x)} + \frac{Bx+C}{(1+x^2)}$

$2 = A(1+x^2) + (Bx+C)(1-x)$

$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$

Equating the coefficient of x^2, x, and constant term, we obtain

$A - B = 0$

$B - C = 0$

$A + C = 2$

On solving these equations, we obtain

$A = 1$, $B = 1$, and $C = 1$

$$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

$$\Rightarrow \int \frac{2}{(1-x)(1+x^2)} dx = \int \frac{1}{1-x} dx + \int \frac{x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= -\int \frac{1}{x-1} dx + \frac{1}{2} \int \frac{2x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= -\log|x-1| + \frac{1}{2} \log|1+x^2| + \tan^{-1} x + C$$

Question 14. $\frac{3x-1}{(x+2)^2}$

Solution :

Let $\frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$

$\Rightarrow 3x-1 = A(x+2)+B$

Equating the coefficient of *x* and constant term, we obtain

A = 3

2*A* + *B* = −1 ⇒ *B* = −7

$$\therefore \frac{3x-1}{(x+2)^2} = \frac{3}{(x+2)} - \frac{7}{(x+2)^2}$$

$$\Rightarrow \int \frac{3x-1}{(x+2)^2}dx = 3\int \frac{1}{(x+2)}dx - 7\int \frac{x}{(x+2)^2}dx$$

$$= 3\log|x+2| - 7\left(\frac{-1}{(x+2)}\right) + C$$

$$= 3\log|x+2| + \frac{7}{(x+2)} + C$$

Question15. $\frac{1}{x^4-1}$

Solution :

$$\frac{1}{(x^4-1)}=\frac{1}{(x^2-1)(x^2+1)}=\frac{1}{(x+1)(x-1)(1+x^2)}$$

$$\text{Let } \frac{1}{(x+1)(x-1)(1+x^2)}=\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{Cx+D}{(x^2+1)}$$

$$1=A(x-1)(x^2+1)+B(x+1)(x^2+1)+(Cx+D)(x^2-1)$$

$$1=A(x^3+x-x^2-1)+B(x^3+x+x^2+1)+Cx^3+Dx^2-Cx-D$$

$$1=(A+B+C)x^3+(-A+B+D)x^2+(A+B-C)x+(-A+B-D)$$

Equating the coefficient of x^3, x^2, x, and constant term, we obtain

$$A+B+C=0$$
$$-A+B+D=0$$
$$A+B-C=0$$
$$-A+B-D=1$$

On solving these equations, we obtain

$$A=-\frac{1}{4}, B=\frac{1}{4}, C=0, \text{ and } D=-\frac{1}{2}$$

$$\therefore \frac{1}{x^4-1}=\frac{-1}{4(x+1)}+\frac{1}{4(x-1)}-\frac{1}{2(x^2+1)}$$

$$\Rightarrow \int\frac{1}{x^4-1}dx=-\frac{1}{4}\log|x-1|+\frac{1}{4}\log|x-1|-\frac{1}{2}\tan^{-1}x+C$$

$$=\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+C$$

Question16. $\frac{1}{x(x^n+1)}$ [Hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$]

Solution :

$$\frac{1}{x(x^n+1)}$$

Multiplying numerator and denominator by x^{n-1}, we obtain

$$\frac{1}{x\left(x^n+1\right)}=\frac{x^{n-1}}{x^{n-1}x\left(x^n+1\right)}=\frac{x^{n-1}}{x^n\left(x^n+1\right)}$$

Let $x^n=t \Rightarrow x^{n-1}dx=dt$

$$\therefore \int\frac{1}{x\left(x^n+1\right)}dx=\int\frac{x^{n-1}}{x^n\left(x^n+1\right)}dx=\frac{1}{n}\int\frac{1}{t(t+1)}dt$$

Let $\frac{1}{t(t+1)}=\frac{A}{t}+\frac{B}{(t+1)}$

$$1=A(1+t)+Bt \qquad ...(1)$$

Substituting t = 0, −1 in equation (1), we obtain

A = 1 and B = −1

$$\therefore \frac{1}{t(t+1)}=\frac{1}{t}-\frac{1}{(1+t)}$$

$$\Rightarrow \int\frac{1}{x\left(x^n+1\right)}dx=\frac{1}{n}\int\left\{\frac{1}{t}-\frac{1}{(t+1)}\right\}dx$$

$$=\frac{1}{n}\left[\log|t|-\log|t+1|\right]+C$$

$$=-\frac{1}{n}\left[\log\left|x^n\right|-\log\left|x^n+1\right|\right]+C$$

$$=\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+C$$

Question17. $\frac{\cos x}{(1-\sin x)(2-\sin x)}$ **[Hint: Put sin x = t]**

Solution :

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$

Let $\sin x = t \Rightarrow \cos x\, dx = dt$

$$\therefore \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int \frac{dt}{(1-t)(2-t)}$$

Let $\frac{1}{(1-t)(2-t)} = \frac{A}{(1-t)} + \frac{B}{(2-t)}$

$$1 = A(2-t) + B(1-t) \qquad \ldots(1)$$

Substituting $t = 2$ and then $t = 1$ in equation (1), we obtain

$A = 1$ and $B = -1$

$$\therefore \frac{1}{(1-t)(2-t)} = \frac{1}{(1-t)} - \frac{1}{(2-t)}$$

$$\Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int\left\{\frac{1}{1-t} - \frac{1}{(2-t)}\right\}dt$$

$$= -\log|1-t| + \log|2-t| + C$$

$$= \log\left|\frac{2-t}{1-t}\right| + C$$

$$= \log\left|\frac{2-\sin x}{1-\sin x}\right| + C$$

Integrate the following function in Exercises 18 to 21.

Question18. $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

Solution :

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \frac{(4x^2+10)}{(x^2+3)(x^2+4)}$$

Let $\frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{Ax+B}{(x^2+3)} + \frac{Cx+D}{(x^2+4)}$

$$4x^2+10 = (Ax+B)(x^2+4) + (Cx+D)(x^2+3)$$

$$4x^2+10 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + 3Cx + Dx^2 + 3D$$

$$4x^2+10 = (A+C)x^3 + (B+D)x^2 + (4A+3C)x + (4B+3D)$$

Equating the coefficients of x^3, x^2, x, and constant term, we obtain

$A + C = 0$

$B + D = 4$

$4A + 3C = 0$

$4B + 3D = 10$

On solving these equations, we obtain

$A = 0$, $B = -2$, $C = 0$, and $D = 6$

$$\therefore \frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}$$

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \left(\frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}\right)$$

$$\Rightarrow \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx = \int \left\{1 + \frac{2}{(x^2+3)} - \frac{6}{(x^2+4)}\right\}dx$$

$$= \int \left\{1 + \frac{2}{x^2+(\sqrt{3})^2} - \frac{6}{x^2+2^2}\right\}$$

$$= x + 2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right) - 6\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right) + C$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\tan^{-1}\frac{x}{2} + C$$

Question19. $\frac{2x}{(x^2+1)(x^2+3)}$

Solution :

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Let $x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore \int \frac{2x}{(x^2+1)(x^2+3)}dx = \int \frac{dt}{(t+1)(t+3)} \qquad ...(1)$$

$$\text{Let } \frac{1}{(t+1)(t+3)} = \frac{A}{(t+1)} + \frac{B}{(t+3)}$$

$$1 = A(t+3) + B(t+1) \qquad ...(1)$$

Substituting $t = -3$ and $t = -1$ in equation (1), we obtain

$$A = \frac{1}{2} \text{ and } B = -\frac{1}{2}$$

$$\therefore \frac{1}{(t+1)(t+3)} = \frac{1}{2(t+1)} - \frac{1}{2(t+3)}$$

$$\Rightarrow \int \frac{2x}{(x^2+1)(x^2+3)}dx = \int \left\{ \frac{1}{2(t+1)} - \frac{1}{2(t+3)} \right\} dt$$

$$= \frac{1}{2}\log|(t+1)| - \frac{1}{2}\log|t+3| + C$$

$$= \frac{1}{2}\log\left|\frac{t+1}{t+3}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$$

Question20. $\frac{1}{x(x^4-1)}$

Solution :

$$\frac{1}{x(x^4-1)}$$

Multiplying numerator and denominator by x^3, we obtain

$$\frac{1}{x\left(x^4-1\right)} = \frac{x^3}{x^4\left(x^4-1\right)}$$

$$\therefore \int \frac{1}{x\left(x^4-1\right)}dx = \int \frac{x^3}{x^4\left(x^4-1\right)}dx$$

Let $x^4 = t \Rightarrow 4x^3dx = dt$

$$\therefore \int \frac{1}{x\left(x^4-1\right)}dx = \frac{1}{4}\int \frac{dt}{t(t-1)}$$

Let $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{(t-1)}$

$1 = A(t-1) + Bt \qquad \text{...(1)}$

Substituting t = 0 and 1 in (1), we obtain

$A = -1$ and $B = 1$

$$\Rightarrow \frac{1}{t(t+1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{x\left(x^4-1\right)}dx = \frac{1}{4}\int \left\{\frac{-1}{t} + \frac{1}{t-1}\right\}dt$$

$$= \frac{1}{4}\left[-\log|t| + \log|t-1|\right] + C$$

$$= \frac{1}{4}\log\left|\frac{t-1}{t}\right| + C$$

$$= \frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right| + C$$

Question21. $\frac{1}{\left(e^x-1\right)}$ **[Hint: Put $e^x = t$]**

Solution :

$$\frac{1}{\left(e^x-1\right)}$$

$$\Rightarrow \int \frac{1}{e^x-1}dx = \int \frac{1}{t-1} \times \frac{dt}{t} = \int \frac{1}{t(t-1)}dt$$

Let $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$

$1 = A(t-1) + Bt \quad \ldots(1)$

Substituting $t = 1$ and $t = 0$ in equation (1), we obtain

$A = -1$ and $B = 1$

$$\therefore \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{t(t-1)}dt = \log\left|\frac{t-1}{t}\right| + C$$

$$= \log\left|\frac{e^x-1}{e^x}\right| + C$$

Choose the correct answer in each of the Exercise 22 and 23.

Question22. $\int \frac{x\,dx}{(x-1)(x-2)}$ **equals:**

A. $\log\left|\frac{(x-1)^2}{x-2}\right| + C$

B. $\log\left|\frac{(x-2)^2}{x-1}\right| + C$

C. $\log\left|\left(\frac{x-1}{x-2}\right)^2\right| + C$

D. log|(x - 1)(x - 2)| + C

Solution :

Let $\frac{x}{(x-1)(x-2)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}$

$x=A(x-2)+B(x-1)$ ...(1)

Substituting x = 1 and 2 in (1), we obtain

A = −1 and B = 2

$$\therefore \frac{x}{(x-1)(x-2)}=-\frac{1}{(x-1)}+\frac{2}{(x-2)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)}dx=\int\left\{\frac{-1}{(x-1)}+\frac{2}{(x-2)}\right\}dx$$

$$=-\log|x-1|+2\log|x-2|+C$$

$$=\log\left|\frac{(x-2)^2}{x-1}\right|+C$$

Therefore, option (B) is correct.

Question23. $\int \frac{dx}{x(x^2+1)}$ **equals:**

A. $\log|x|-\frac{1}{2}\log(x^2+1)+C$

B. $\log|x|+\frac{1}{2}\log(x^2+1)+C$

C. $-\log|x|+\frac{1}{2}\log(x^2+1)+C$

D. $\frac{1}{2}\log|x|+\log(x^2+1)+C$

Solution **:**

$$\text{Let } \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + (Bx+C)x$$

Equating the coefficients of x^2, x, and constant term, we obtain

$A + B = 0$

$C = 0$

$A = 1$

On solving these equations, we obtain

$A = 1$, $B = -1$, and $C = 0$

$$\therefore \frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1}$$

$$\Rightarrow \int \frac{1}{x(x^2+1)} dx = \int \left\{ \frac{1}{x} - \frac{x}{x^2+1} \right\} dx$$

$$= \log|x| - \frac{1}{2}\log|x^2+1| + C$$

Therefore, option (A) is correct.

Exercise 7.5 Page: 263

Question 1. *x* sin *x*

Solution : Let I = $\int x \sin x \, dx$

Taking x as first function and sin x as second function and integrating by parts, we obtain

$$I = x\int \sin x \, dx - \int \left\{ \left(\frac{d}{dx} x \right) \int \sin x \, dx \right\} dx$$

$$= x(-\cos x) - \int 1 \cdot (-\cos x) dx$$

$$= -x\cos x + \sin x + C$$

Question2. *x* sin3*x*

Solution **:**

Let I = $\int x \sin 3x \, dx$

Taking x as first function and sin $3x$ as second function and integrating by parts, we obtain

$$I = x\int \sin 3x\,dx - \int\left\{\left(\frac{d}{dx}x\right)\int \sin 3x\,dx\right\}$$
$$= x\left(\frac{-\cos 3x}{3}\right) - \int 1\cdot\left(\frac{-\cos 3x}{3}\right)dx$$
$$= \frac{-x\cos 3x}{3} + \frac{1}{3}\int \cos 3x\,dx$$
$$= \frac{-x\cos 3x}{3} + \frac{1}{9}\sin 3x + C$$

Question3. $x^2 e^x$
Solution :

Let I = $\int x^2 e^x\,dx$

Taking x^2 as first function and e^x as second function and integrating by parts, we obtain

$$I = x^2\int e^x\,dx - \int\left\{\left(\frac{d}{dx}x^2\right)\int e^x dx\right\}dx$$
$$= x^2e^x - \int 2x\cdot e^x dx$$
$$= x^2e^x - 2\int x\cdot e^x dx$$

Again integrating by parts, we obtain

$$= x^2e^x - 2\left[x\cdot\int e^x dx - \int\left\{\left(\frac{d}{dx}x\right)\cdot\int e^x dx\right\}dx\right]$$
$$= x^2e^x - 2\left[xe^x - \int e^x dx\right]$$
$$= x^2e^x - 2\left[xe^x - e^x\right]$$
$$= x^2e^x - 2xe^x + 2e^x + C$$
$$= e^x\left(x^2 - 2x + 2\right) + C$$

Question 4. $x\log x$

Solution : Let I = $\int x \log x\,dx$

Taking log x as first function and x as second function and integrating by parts, we obtain

$$I = \log x \int x\ dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x\ dx \right\} dx$$
$$= \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$$
$$= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx$$
$$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$$

Question 5. ***x* log 2*x***

Solution : *Let* $I = \int x \log 2x\ dx$

Taking log 2*x* as first function and *x* as second function and integrating by parts, we obtain

$$I = \log 2x \int x\ dx - \int \left\{ \left(\frac{d}{dx} 2 \log x \right) \int x\ dx \right\} dx$$
$$= \log 2x \cdot \frac{x^2}{2} - \int \frac{2}{2x} \cdot \frac{x^2}{2} dx$$
$$= \frac{x^2 \log 2x}{2} - \int \frac{x}{2} dx$$
$$= \frac{x^2 \log 2x}{2} - \frac{x^2}{4} + C$$

Question 6. $x^2 \log x$

Solution : Let $I = \int x^2 \log x\ dx$

Taking log x as first function and x^2 as second function and integrating by parts, we obtain

$$I = \log x \int x^2\ dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x^2 dx \right\} dx$$
$$= \log x \left(\frac{x^3}{3} \right) - \int \frac{1}{x} \cdot \frac{x^3}{3} dx$$
$$= \frac{x^3 \log x}{3} - \int \frac{x^2}{3} dx$$
$$= \frac{x^3 \log x}{3} - \frac{x^3}{9} + C$$

Question7. xsin^{-1} x
Solution :
Let I = ∫ x sin^{-1} x

Taking **sin^{-1} x** as first function and *x* as second function and integrating by parts, we obtain

$$I=\sin^{-1}x\int x\,dx-\int\left\{\left(\frac{d}{dx}\sin^{-1}x\right)\int x\,dx\right\}dx$$

$$=\sin^{-1}x\left(\frac{x^2}{2}\right)-\int\frac{1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\int\left\{\frac{1-x^2}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\int\left\{\sqrt{1-x^2}-\frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\left\{\int\sqrt{1-x^2}\,dx-\int\frac{1}{\sqrt{1-x^2}}dx\right\}$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\left\{\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x-\sin^{-1}x\right\}+C$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{x}{4}\sqrt{1-x^2}+\frac{1}{4}\sin^{-1}x-\frac{1}{2}\sin^{-1}x+C$$

$$=\frac{1}{4}\left(2x^2-1\right)\sin^{-1}x+\frac{x}{4}\sqrt{1-x^2}+C$$

Question8. xtan^{-1} x
Solution :
Let I = ∫ x tan^{-1} x

Taking tan^{-1} x as first function and *x* as second function and integrating by parts, we obtain

$$I = \tan^{-1} x \int x\,dx - \int\left\{\left(\frac{d}{dx}\tan^{-1} x\right)\int x\,dx\right\}dx$$

$$= \tan^{-1} x\left(\frac{x^2}{2}\right) - \int\frac{1}{1+x^2}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int\frac{x^2}{1+x^2}\,dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int\left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\left(x - \tan^{-1} x\right) + C$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{x}{2} + \frac{1}{2}\tan^{-1} x + C$$

Integrate the functions in Exercises 9 to 15.
Question 9. x $\cos^{-1}$ x
Solution :
Let I = ∫ **x $\cos^{-1}$ x**

Taking $\cos^{-1} x$ as first function and x as second function and integrating by parts, we obtain

$$I = \cos^{-1} x \int x\,dx - \int \left\{ \left(\frac{d}{dx} \cos^{-1} x \right) \int x\,dx \right\} dx$$

$$= \cos^{-1} x \frac{x^2}{2} - \int \frac{-1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \left\{ \sqrt{1-x^2} + \left(\frac{-1}{\sqrt{1-x^2}} \right) \right\} dx$$

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \sqrt{1-x^2}\,dx - \frac{1}{2} \int \left(\frac{-1}{\sqrt{1-x^2}} \right) dx$$

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} I_1 - \frac{1}{2} \cos^{-1} x \qquad \ldots(1)$$

where, $I_1 = \int \sqrt{1-x^2}\,dx$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{d}{dx} \sqrt{1-x^2} \int x\,dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{-2x}{2\sqrt{1-x^2}} . x\,dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}} dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \left\{ \int \sqrt{1-x^2}\,dx + \int \frac{-dx}{\sqrt{1-x^2}} \right\}$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \left\{ I_1 + \cos^{-1} x \right\}$$

$$\Rightarrow 2I_1 = x\sqrt{1-x^2} - \cos^{-1} x$$

$$\therefore I_1 = \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2} \cos^{-1} x$$

Substituting in (1), we obtain

$$I = \frac{x \cos^{-1} x}{2} - \frac{1}{2} \left(\frac{x}{2}\sqrt{1-x^2} - \frac{1}{2} \cos^{-1} x \right) - \frac{1}{2} \cos^{-1} x$$

$$= \frac{(2x^2-1)}{4} \cos^{-1} x - \frac{x}{4}\sqrt{1-x^2} + C$$

Question10. $(\sin^{-1} x)^2$

Solution :

Let I = ∫ $(\sin^{-1} x)^2$.1 dx

Taking $(\sin^{-1}x)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$I=\left(\sin^{-1}x\right)\int 1\,dx-\int\left\{\frac{d}{dx}\left(\sin^{-1}x\right)^2\cdot\int 1\cdot dx\right\}dx$$

$$=\left(\sin^{-1}x\right)^2\cdot x-\int\frac{2\sin^{-1}x}{\sqrt{1-x^2}}\cdot x\,dx$$

$$=x\left(\sin^{-1}x\right)^2+\int\sin^{-1}x\cdot\left(\frac{-2x}{\sqrt{1-x^2}}\right)dx$$

$$=x\left(\sin^{-1}x\right)^2+\left[\sin^{-1}x\int\frac{-2x}{\sqrt{1-x^2}}dx-\int\left\{\left(\frac{d}{dx}\sin^{-1}x\right)\int\frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$$

$$=x\left(\sin^{-1}x\right)^2+\left[\sin^{-1}x\cdot 2\sqrt{1-x^2}-\int\frac{1}{\sqrt{1-x^2}}\cdot 2\sqrt{1-x^2}\,dx\right]$$

$$=x\left(\sin^{-1}x\right)^2+2\sqrt{1-x^2}\sin^{-1}x-\int 2\,dx$$

$$=x\left(\sin^{-1}x\right)^2+2\sqrt{1-x^2}\sin^{-1}x-2x+C$$

Question11. $\frac{x\cos^{-1}x}{\sqrt{1-x^2}}$

Solution **:**

Let $I=\int\frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

$$I=\frac{-1}{2}\int\frac{-2x}{\sqrt{1-x^2}}\cdot\cos^{-1}x\,dx$$

Taking $\cos^{-1}x$ as first function and $\left(\frac{-2x}{\sqrt{1-x^2}}\right)$ as second function and integrating by parts, we obtain

$$I=\frac{-1}{2}\left[\cos^{-1}x\int\frac{-2x}{\sqrt{1-x^2}}dx-\int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int\frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$$
$$=\frac{-1}{2}\left[\cos^{-1}x\cdot 2\sqrt{1-x^2}-\int\frac{-1}{\sqrt{1-x^2}}\cdot 2\sqrt{1-x^2}\,dx\right]$$
$$=\frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x+\int 2\,dx\right]$$
$$=\frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x+2x\right]+C$$
$$=-\left[\sqrt{1-x^2}\cos^{-1}x+x\right]+C$$

Question12. $x\sec^2 x$

Solution : Let $I = \int x \sec^2 x\, dx$

Taking *x* as first function and $\sec^2 x$ as second function and integrating by parts, we obtain

$$I=x\int\sec^2 x\,dx-\int\left\{\left\{\frac{d}{dx}x\right\}\int\sec^2 x\,dx\right\}dx$$
$$=x\tan x-\int 1\cdot\tan x\,dx$$
$$=x\tan x+\log|\cos x|+C$$

Question 13. $\tan^{-1} x$

Solution :

Let $I = \int \tan^{-1} x\, dx$

Taking $\tan^{-1} x$ as first function and 1 as second function and integrating by parts, we obtain

$$I=\tan^{-1}x\int 1dx-\int\left\{\left(\frac{d}{dx}\tan^{-1}x\right)\int 1\cdot dx\right\}dx$$
$$=\tan^{-1}x\cdot x-\int\frac{1}{1+x^2}\cdot x\,dx$$
$$=x\tan^{-1}x-\frac{1}{2}\int\frac{2x}{1+x^2}dx$$
$$=x\tan^{-1}x-\frac{1}{2}\log\left|1+x^2\right|+C$$
$$=x\tan^{-1}x-\frac{1}{2}\log\left(1+x^2\right)+C$$

Question14. $x(\log x)^2$

Solution :

Let $I = \int x (\log x)^2\, dx$

Taking $(\log x)^2$ as first function and x as second function and integrating by parts, we obtain

$$I = (\log x)^2 \int x dx - \int \left[\left\{\frac{d}{dx}(\log x)^2\right\} \int x dx\right] dx$$

$$= \frac{x^2}{2}(\log x)^2 - \left[\int 2 \log x \cdot \frac{1}{x} \cdot \frac{x^2}{2} dx\right]$$

$$= \frac{x^2}{2}(\log x)^2 - \int x \log x \, dx$$

Again integrating by parts, we obtain

$$I = \frac{x^2}{2}(\log x)^2 - \left[\log x \int x \, dx - \int \left\{\left(\frac{d}{dx}\log x\right) \int x \, dx\right\} dx\right]$$

$$= \frac{x^2}{2}(\log x)^2 - \left[\frac{x^2}{2}\log x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx\right]$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{1}{2}\int x \, dx$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4} + C$$

Question 15. $(x^2 + 1) \log x$

Solution **:**

Let $I = \int(x^2+1)\log x\,dx = \int x^2 \log x\,dx + \int \log x\,dx$

Let $I = I_1 + I_2$ … (1)

Where, $I_1 = \int x^2 \log x\,dx$ and $I_2 = \int \log x\,dx$

$I_1 = \int x^2 \log x dx$

Taking log x as first function and x^2 as second function and integrating by parts, we obtain

$$\begin{aligned} I_1 &= \log x - \int x^2 dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int x^2 dx\right\}dx \\ &= \log x \cdot \frac{x^3}{3} - \int \frac{1}{x}\cdot\frac{x^3}{3}dx \\ &= \frac{x^3}{3}\log x - \frac{1}{3}\left(\int x^2 dx\right) \\ &= \frac{x^3}{3}\log x - \frac{x^3}{9} + C_1 \qquad \dots(2) \end{aligned}$$

$I_2 = \int \log x\,dx$

Taking log x as first function and 1 as second function and integrating by parts, we obtain

$$\begin{aligned} I_2 &= \log x\int 1\cdot dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int 1\cdot dx\right\} \\ &= \log x \cdot x - \int \frac{1}{x}\cdot x dx \\ &= x\log x - \int 1 dx \\ &= x\log x - x + C_2 \qquad \dots(3) \end{aligned}$$

Using equations (2) and (3) in (1), we obtain

$$\begin{aligned} I &= \frac{x^3}{3}\log x - \frac{x^3}{9} + C_1 + x\log x - x + C_2 \\ &= \frac{x^3}{3}\log x - \frac{x^3}{9} + x\log x - x + (C_1 + C_2) \\ &= \left(\frac{x^3}{3} + x\right)\log x - \frac{x^3}{9} - x + C \end{aligned}$$

Integrate the functions in Exercises 16 to 22.

Question16. $e^x(\sin x + \cos x)$

Solution **:**

Let $I = \int e^x (\sin x + \cos x) dx$

Let $f(x) = \sin x$

$\Rightarrow f'(x) = \cos x$

$\therefore I = \int e^x \{f(x) + f'(x)\} dx$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

$\therefore I = e^x \sin x + C$

Question17. $\dfrac{xe^x}{(1+x)^2}$

Solution :

Let $I = \int \frac{xe^x}{(1+x)^2} dx = \int e^x \left\{ \frac{x}{(1+x)^2} \right\} dx$

$= \int e^x \left\{ \frac{1+x-1}{(1+x)^2} \right\} dx$

$= \int e^x \left\{ \frac{1}{1+x} - \frac{1}{(1+x)^2} \right\} dx$

Let $f(x) = \frac{1}{1+x} \Rightarrow f'(x) = \frac{-1}{(1+x)^2}$

$\Rightarrow \int \frac{xe^x}{(1+x)^2} dx = \int e^x \{f(x) + f'(x)\} dx$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

$\therefore \int \frac{xe^x}{(1+x)^2} dx = \frac{e^x}{1+x} + C$

Question18. $e^x \dfrac{1+\sin x}{1+\cos x}$

Solution :

$$e^x\left(\frac{1+\sin x}{1+\cos x}\right)$$

$$=e^x\left(\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$$

$$=\frac{e^x\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}$$

$$=\frac{1}{2}e^x\cdot\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}}\right)^2$$

$$=\frac{1}{2}e^x\left[\tan\frac{x}{2}+1\right]^2$$

$$=\frac{1}{2}e^2\left(1+\tan\frac{x}{2}\right)^2$$

$$=\frac{1}{2}e^x\left[1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}\right]$$

$$=\frac{1}{2}e^x\left[\sec^2\frac{x}{2}+2\tan\frac{x}{2}\right]$$

$$\frac{e^x(1+\sin x)dx}{(1+\cos x)}=e^x\left[\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right] \qquad ...(1)$$

Let $\tan\frac{x}{2}=f(x) \Rightarrow f'(x)=\frac{1}{2}\sec^2\frac{x}{2}$

It is known that, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$

From equation (1), we obtain

$$\int\frac{e^x(1+\sin x)}{(1+\cos x)}dx=e^x\tan\frac{x}{2}+C$$

Question19. $e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$

Solution :

Let $I = \int e^x \left[\frac{1}{x} - \frac{1}{x^2}\right] dx$

Also, let $\frac{1}{x} = f(x) \Rightarrow f'(x) = \frac{-1}{x^2}$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

$\therefore I = \frac{e^x}{x} + C$

Question20. $\frac{(x-3)e^x}{(x-1)^3}$

Solution **:**

$$\int e^x \left\{\frac{x-3}{(x-1)^3}\right\} dx = \int e^x \left\{\frac{x-1-2}{(x-1)^3}\right\} dx$$

$$= \int e^x \left\{\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right\} dx$$

Let $f(x) = \frac{1}{(x-1)^2} \Rightarrow f'(x) = \frac{-2}{(x-1)^3}$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

$$\therefore \int e^x \left\{\frac{(x-3)}{(x-1)^2}\right\} dx = \frac{e^x}{(x-1)^2} + C$$

Question21. e^{2x} **sinx**

Solution **:**

Let I = $\int e^{2x} \sin x$

Integrating by parts, we obtain

$$I = \sin x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \sin x \right) \int e^{2x} dx \right\} dx$$

$$\Rightarrow I = \sin x \cdot \frac{e^{2x}}{2} - \int \cos x \cdot \frac{e^{2x}}{2} dx$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \int e^{2x} \cos x \, dx$$

Again integrating by parts, we obtain

$$I = \frac{e^{2x} \cdot \sin x}{2} - \frac{1}{2} \left[\cos x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \cos x \right) \int e^{2x} dx \right\} dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \left[\cos x \cdot \frac{e^{2x}}{2} - \int (-\sin x) \frac{e^{2x}}{2} dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \cdot \sin x}{2} - \frac{1}{2} \left[\frac{e^{2x} \cos x}{2} + \frac{1}{2} \int e^{2x} \sin x dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{4} I \qquad \text{[From (1)]}$$

$$\Rightarrow I + \frac{1}{4} I = \frac{e^{2x} \cdot \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$\Rightarrow \frac{5}{4} I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$\Rightarrow I = \frac{4}{5} \left[\frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} \right] + C$$

$$\Rightarrow I = \frac{e^{2x}}{5} [2 \sin x - \cos x] + C$$

Question22. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Solution :

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$\Rightarrow \int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int 2\theta\cdot\sec^2\theta\, d\theta = 2\int \theta\cdot\sec^2\theta\, d\theta$$

Integrating by parts, we obtain

$$2\left[\theta\cdot\int\sec^2\theta d\theta - \int\left\{\left(\frac{d}{d\theta}\theta\right)\int\sec^2\theta d\theta\right\}d\theta\right]$$

$$= 2\left[\theta\cdot\tan\theta - \int\tan\theta d\theta\right]$$

$$= 2\left[\theta\tan\theta + \log|\cos\theta|\right] + C$$

$$= 2\left[x\tan^{-1}x + \log\left|\frac{1}{\sqrt{1+x^2}}\right|\right] + C$$

$$= 2x\tan^{-1}x + 2\log(1+x^2)^{-\frac{1}{2}} + C$$

$$= 2x\tan^{-1}x + 2\left[-\frac{1}{2}\log(1+x^2)\right] + C$$

$$= 2x\tan^{-1}x - \log(1+x^2) + C$$

Choose the correct answer in Exercise 23 and 24.

Question23. $\int x^2 e^{x^3} dx$ **equalsto**

(A) $\frac{1}{3}e^{x^3} + C$

(B) $\frac{1}{3}e^{x^2} + C$

(C) $\frac{1}{2}e^{x^3} + C$

(D) $\frac{1}{3}e^{x^2} + C$

Solution :

LetI= $\int x^2 e^{x^3} dx$

Also, let $x^3 = t \Rightarrow 3x^2 dx = dt$

$$\Rightarrow I = \frac{1}{3}\int e^t dt$$
$$= \frac{1}{3}\left(e^t\right) + C$$
$$= \frac{1}{3}e^{x^3} + C$$

Therefore, option (A) is correct.

Question24. $\int e^x \sec x(1 + \tan x)dx$ **equals:**

(A) e^{x}cosx+C
(B) e^{x}secx+C
(C) e^x sinx+C
(D) e^{x}tanx+C
Solution :

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$\Rightarrow \int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int 2\theta\cdot\sec^2\theta\, d\theta = 2\int \theta\cdot\sec^2\theta\, d\theta$$

Integrating by parts, we obtain

$$2\left[\theta\cdot\int\sec^2\theta d\theta - \int\left\{\left(\frac{d}{d\theta}\theta\right)\int\sec^2\theta d\theta\right\}d\theta\right]$$

$$= 2\left[\theta\cdot\tan\theta - \int\tan\theta d\theta\right]$$

$$= 2\left[\theta\tan\theta + \log|\cos\theta|\right] + C$$

$$= 2\left[x\tan^{-1}x + \log\left|\frac{1}{\sqrt{1+x^2}}\right|\right] + C$$

$$= 2x\tan^{-1}x + 2\log(1+x^2)^{-\frac{1}{2}} + C$$

$$= 2x\tan^{-1}x + 2\left[-\frac{1}{2}\log(1+x^2)\right] + C$$

$$= 2x\tan^{-1}x - \log(1+x^2) + C$$

Therefore, option (B) is correct.

Let $I = \int e^x \sec x(1+\tan x)dx = \int e^x(\sec x + \sec x\tan x)dx$

Also, let $\sec x = f(x) \Rightarrow \sec x\tan x = f'(x)$

It is known that, $\int e^x\{f(x)+f'(x)\}dx = e^x f(x) + C$

$$\therefore I = e^x\sec x + C$$

Also, let $x^3 = t \Rightarrow 3x^2dx = dt$

$$\Rightarrow I = \frac{1}{3}\int e^t dt$$

$$= \frac{1}{3}(e^t) + C$$

$$= \frac{1}{3}e^{x^3} + C$$

Exercise 7.6 Page: 263

Question 1. ***x* sin *x***

Solution : Let I = $\int x\sin x\, dx$

Taking x as first function and $\sin x$ as second function and integrating by parts, we obtain

$$I = x\int \sin x\ dx - \int\left\{\left(\frac{d}{dx}x\right)\int \sin x\ dx\right\}dx$$
$$= x(-\cos x) - \int 1\cdot(-\cos x)dx$$
$$= -x\cos x + \sin x + C$$

Question2. *x* sin3*x*
Solution **:**
Let I = ∫ *x* sin *3x dx*

Taking *x* as first function and sin 3*x* as second function and integrating by parts, we obtain

$$I = x\int \sin 3x\ dx - \int\left\{\left(\frac{d}{dx}x\right)\int \sin 3x\ dx\right\}$$
$$= x\left(\frac{-\cos 3x}{3}\right) - \int 1\cdot\left(\frac{-\cos 3x}{3}\right)dx$$
$$= \frac{-x\cos 3x}{3} + \frac{1}{3}\int \cos 3x\ dx$$
$$= \frac{-x\cos 3x}{3} + \frac{1}{9}\sin 3x + C$$

Question3. $x^2 e^x$
Solution :

Let I = $\int x^2 e^x\, dx$

Taking x^2 as first function and e^x as second function and integrating by parts, we obtain

$$I = x^2\int e^x\ dx - \int\left\{\left(\frac{d}{dx}x^2\right)\int e^x dx\right\}dx$$
$$= x^2e^x - \int 2x\cdot e^x dx$$
$$= x^2e^x - 2\int x\cdot e^x dx$$

Again integrating by parts, we obtain

$$= x^2e^x - 2\left[x \cdot \int e^x dx - \int\left\{\left(\frac{d}{dx}x\right) \cdot \int e^x dx\right\}dx\right]$$
$$= x^2e^x - 2\left[xe^x - \int e^x dx\right]$$
$$= x^2e^x - 2\left[xe^x - e^x\right]$$
$$= x^2e^x - 2xe^x + 2e^x + C$$
$$= e^x\left(x^2 - 2x + 2\right) + C$$

Question 4. ***x* log*x***

Solution : Let I = ∫ *x* log*x dx*

Taking log *x* as first function and *x* as second function and integrating by parts, we obtain

$$I = \log x \int x\ dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int x\ dx\right\}dx$$
$$= \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$$
$$= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx$$
$$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$$

Question 5. ***x* log 2*x***

Solution : *Let I* = ∫ *x* log 2*x dx*

Taking log 2*x* as first function and *x* as second function and integrating by parts, we obtain

$$I = \log 2x \int x\ dx - \int\left\{\left(\frac{d}{dx}2\log x\right)\int x\ dx\right\}dx$$
$$= \log 2x \cdot \frac{x^2}{2} - \int \frac{2}{2x} \cdot \frac{x^2}{2} dx$$
$$= \frac{x^2 \log 2x}{2} - \int \frac{x}{2} dx$$
$$= \frac{x^2 \log 2x}{2} - \frac{x^2}{4} + C$$

Question 6. x^2 log *x*

Solution : Let I = ∫ x^2 log x dx

Taking log x as first function and x^2 as second function and integrating by parts, we obtain

$$I = \log x \int x^2\,dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int x^2 dx\right\}dx$$
$$= \log x\left(\frac{x^3}{3}\right) - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx$$
$$= \frac{x^3 \log x}{3} - \int \frac{x^2}{3}\,dx$$
$$= \frac{x^3 \log x}{3} - \frac{x^3}{9} + C$$

Question7. $x\sin^{-1} x$

Solution **:**

Let I = $\int x \sin^{-1} x$

Taking **$\sin^{-1} x$** as first function and *x* as second function and integrating by parts, we obtain

$$I = \sin^{-1} x \int x\,dx - \int\left\{\left(\frac{d}{dx}\sin^{-1} x\right)\int x\,dx\right\}dx$$
$$= \sin^{-1} x\left(\frac{x^2}{2}\right) - \int \frac{1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}\,dx$$
$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\int \frac{-x^2}{\sqrt{1-x^2}}\,dx$$
$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\int\left\{\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}\right\}dx$$
$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\int\left\{\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}}\right\}dx$$
$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\left\{\int\sqrt{1-x^2}\,dx - \int\frac{1}{\sqrt{1-x^2}}\,dx\right\}$$
$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\left\{\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x - \sin^{-1} x\right\} + C$$
$$= \frac{x^2 \sin^{-1} x}{2} + \frac{x}{4}\sqrt{1-x^2} + \frac{1}{4}\sin^{-1} x - \frac{1}{2}\sin^{-1} x + C$$
$$= \frac{1}{4}\left(2x^2-1\right)\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + C$$

Question8. $x\tan^{-1} x$

Solution **:**

Let I = $\int x \tan^{-1} x$

Taking $\tan^{-1} x$ as first function and *x* as second function and integrating by parts, we obtain

$$I = \tan^{-1} x \int x\,dx - \int\left\{\left(\frac{d}{dx}\tan^{-1} x\right)\int x\,dx\right\}dx$$

$$= \tan^{-1} x\left(\frac{x^2}{2}\right) - \int\frac{1}{1+x^2}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{x^2\tan^{-1} x}{2} - \frac{1}{2}\int\frac{x^2}{1+x^2}\,dx$$

$$= \frac{x^2\tan^{-1} x}{2} - \frac{1}{2}\int\left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2\tan^{-1} x}{2} - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2\tan^{-1} x}{2} - \frac{1}{2}\left(x - \tan^{-1} x\right) + C$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{x}{2} + \frac{1}{2}\tan^{-1} x + C$$

Integrate the functions in Exercises 9 to 15.
Question 9. $x \cos^{-1} x$
Solution :
Let I = $\int$ **$x \cos^{-1} x$**

Taking $\cos^{-1} x$ as first function and x as second function and integrating by parts, we obtain

$$I=\cos^{-1}x\int x\,dx-\int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int x\,dx\right\}dx$$

$$=\cos^{-1}x\frac{x^2}{2}-\int\frac{-1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\left\{\sqrt{1-x^2}+\left(\frac{-1}{\sqrt{1-x^2}}\right)\right\}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\sqrt{1-x^2}dx-\frac{1}{2}\int\left(\frac{-1}{\sqrt{1-x^2}}\right)dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}I_1-\frac{1}{2}\cos^{-1}x \qquad \ldots(1)$$

where, $I_1=\int\sqrt{1-x^2}\,dx$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{d}{dx}\sqrt{1-x^2}\int x\,dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{-2x}{2\sqrt{1-x^2}}.x\,dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\left\{\int\sqrt{1-x^2}\,dx+\int\frac{-dx}{\sqrt{1-x^2}}\right\}$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\left\{I_1+\cos^{-1}x\right\}$$

$$\Rightarrow 2I_1=x\sqrt{1-x^2}-\cos^{-1}x$$

$$\therefore I_1=\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\cos^{-1}x$$

Substituting in (1), we obtain

$$I=\frac{x\cos^{-1}x}{2}-\frac{1}{2}\left(\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\cos^{-1}x\right)-\frac{1}{2}\cos^{-1}x$$

$$=\frac{(2x^2-1)}{4}\cos^{-1}x-\frac{x}{4}\sqrt{1-x^2}+C$$

Question10. $(\sin^{-1}x)^2$

Solution :

Let I = $\int(\sin^{-1}x)^2$**.**1 dx

Taking $(\sin^{-1}x)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$I = (\sin^{-1}x)\int 1\,dx - \int\left\{\frac{d}{dx}(\sin^{-1}x)^2 \cdot \int 1\cdot dx\right\}dx$$
$$= (\sin^{-1}x)^2 \cdot x - \int \frac{2\sin^{-1}x}{\sqrt{1-x^2}}\cdot x\,dx$$
$$= x(\sin^{-1}x)^2 + \int \sin^{-1}x \cdot \left(\frac{-2x}{\sqrt{1-x^2}}\right)dx$$
$$= x(\sin^{-1}x)^2 + \left[\sin^{-1}x\int\frac{-2x}{\sqrt{1-x^2}}dx - \int\left\{\left(\frac{d}{dx}\sin^{-1}x\right)\int\frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$$
$$= x(\sin^{-1}x)^2 + \left[\sin^{-1}x \cdot 2\sqrt{1-x^2} - \int\frac{1}{\sqrt{1-x^2}}\cdot 2\sqrt{1-x^2}\,dx\right]$$
$$= x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\sin^{-1}x - \int 2\,dx$$
$$= x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\sin^{-1}x - 2x + C$$

Question11. $\dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}$

Solution **:**

Let $I = \int\frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

$$I = \frac{-1}{2}\int\frac{-2x}{\sqrt{1-x^2}}\cdot\cos^{-1}x\,dx$$

Taking $\cos^{-1}x$ as first function and $\left(\frac{-2x}{\sqrt{1-x^2}}\right)$ as second function and integrating by parts, we obtain

$$I = \frac{-1}{2}\left[\cos^{-1}x\int\frac{-2x}{\sqrt{1-x^2}}dx - \int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int\frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$$
$$= \frac{-1}{2}\left[\cos^{-1}x\cdot 2\sqrt{1-x^2} - \int\frac{-1}{\sqrt{1-x^2}}\cdot 2\sqrt{1-x^2}\,dx\right]$$
$$= \frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x + \int 2\,dx\right]$$
$$= \frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x + 2x\right] + C$$
$$= -\left[\sqrt{1-x^2}\cos^{-1}x + x\right] + C$$

Question12. $x\sec^2 x$
Solution : Let $I = \int x \sec^2 x\, dx$

Taking x as first function and $\sec^2 x$ as second function and integrating by parts, we obtain

$$I = x\int\sec^2 x\,dx - \int\left\{\left\{\frac{d}{dx}x\right\}\int\sec^2 x\,dx\right\}dx$$
$$= x\tan x - \int 1\cdot\tan x\,dx$$
$$= x\tan x + \log|\cos x| + C$$

Question 13. $\tan^{-1} x$

Solution :
Let $I = \int \tan^{-1} x\, dx$

Taking $\tan^{-1} x$ as first function and 1 as second function and integrating by parts, we obtain

$$I = \tan^{-1}x\int 1dx - \int\left\{\left(\frac{d}{dx}\tan^{-1}x\right)\int 1\cdot dx\right\}dx$$
$$= \tan^{-1}x\cdot x - \int\frac{1}{1+x^2}\cdot x\,dx$$
$$= x\tan^{-1}x - \frac{1}{2}\int\frac{2x}{1+x^2}dx$$
$$= x\tan^{-1}x - \frac{1}{2}\log\left|1+x^2\right| + C$$
$$= x\tan^{-1}x - \frac{1}{2}\log\left(1+x^2\right) + C$$

Question14. $x(\log x)^2$
Solution :
Let $I = \int x (\log x)^2 dx$

Taking $(\log x)^2$ as first function and x as second function and integrating by parts, we obtain

$$I = (\log x)^2 \int x dx - \int \left[\left\{\frac{d}{dx}(\log x)^2\right\} \int x dx\right] dx$$
$$= \frac{x^2}{2}(\log x)^2 - \left[\int 2\log x \cdot \frac{1}{x} \cdot \frac{x^2}{2} dx\right]$$
$$= \frac{x^2}{2}(\log x)^2 - \int x \log x \, dx$$

Again integrating by parts, we obtain

$$I = \frac{x^2}{2}(\log x)^2 - \left[\log x \int x \, dx - \int \left\{\left(\frac{d}{dx}\log x\right) \int x \, dx\right\} dx\right]$$
$$= \frac{x^2}{2}(\log x)^2 - \left[\frac{x^2}{2}\log x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx\right]$$
$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{1}{2}\int x \, dx$$
$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4} + C$$

Question 15. $(x^2 + 1) \log x$

Solution **:**

Let $I = \int (x^2+1)\log x\,dx = \int x^2 \log x\,dx + \int \log x\,dx$

Let $I = I_1 + I_2$ … (1)

Where, $I_1 = \int x^2 \log x\,dx$ and $I_2 = \int \log x\,dx$

$I_1 = \int x^2 \log x dx$

Taking log x as first function and x^2 as second function and integrating by parts, we obtain

$$\begin{aligned} I_1 &= \log x - \int x^2 dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int x^2 dx\right\}dx \\ &= \log x \cdot \frac{x^3}{3} - \int \frac{1}{x}\cdot\frac{x^3}{3}dx \\ &= \frac{x^3}{3}\log x - \frac{1}{3}\left(\int x^2 dx\right) \\ &= \frac{x^3}{3}\log x - \frac{x^3}{9} + C_1 \qquad \ldots(2) \end{aligned}$$

$I_2 = \int \log x\,dx$

Taking log x as first function and 1 as second function and integrating by parts, we obtain

$$\begin{aligned} I_2 &= \log x \int 1\cdot dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int 1\cdot dx\right\} \\ &= \log x \cdot x - \int \frac{1}{x}\cdot x dx \\ &= x\log x - \int 1 dx \\ &= x\log x - x + C_2 \qquad \ldots(3) \end{aligned}$$

Using equations (2) and (3) in (1), we obtain

$$\begin{aligned} I &= \frac{x^3}{3}\log x - \frac{x^3}{9} + C_1 + x\log x - x + C_2 \\ &= \frac{x^3}{3}\log x - \frac{x^3}{9} + x\log x - x + (C_1 + C_2) \\ &= \left(\frac{x^3}{3} + x\right)\log x - \frac{x^3}{9} - x + C \end{aligned}$$

Integrate the functions in Exercises 16 to 22.

Question16. $e^x(\sin x + \cos x)$

Solution **:**

Let $I = \int e^x (\sin x + \cos x) dx$

Let $f(x) = \sin x$

$\Rightarrow f'(x) = \cos x$

$\therefore I = \int e^x \{f(x) + f'(x)\} dx$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

$\therefore I = e^x \sin x + C$

Question17. $\dfrac{xe^x}{(1+x)^2}$

Solution :

Let $I = \int \frac{xe^x}{(1+x)^2} dx = \int e^x \left\{ \frac{x}{(1+x)^2} \right\} dx$

$= \int e^x \left\{ \frac{1+x-1}{(1+x)^2} \right\} dx$

$= \int e^x \left\{ \frac{1}{1+x} - \frac{1}{(1+x)^2} \right\} dx$

Let $f(x) = \frac{1}{1+x} \Rightarrow f'(x) = \frac{-1}{(1+x)^2}$

$\Rightarrow \int \frac{xe^x}{(1+x)^2} dx = \int e^x \{f(x) + f'(x)\} dx$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

$\therefore \int \frac{xe^x}{(1+x)^2} dx = \frac{e^x}{1+x} + C$

Question18. $e^x \dfrac{1+\sin x}{1+\cos x}$

Solution :

$$e^x\left(\frac{1+\sin x}{1+\cos x}\right)$$

$$=e^x\left(\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$$

$$=\frac{e^x\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}$$

$$=\frac{1}{2}e^x\cdot\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}}\right)^2$$

$$=\frac{1}{2}e^x\left[\tan\frac{x}{2}+1\right]^2$$

$$=\frac{1}{2}e^2\left(1+\tan\frac{x}{2}\right)^2$$

$$=\frac{1}{2}e^x\left[1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}\right]$$

$$=\frac{1}{2}e^x\left[\sec^2\frac{x}{2}+2\tan\frac{x}{2}\right]$$

$$\frac{e^x(1+\sin x)dx}{(1+\cos x)}=e^x\left[\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right] \qquad ...(1)$$

Let $\tan\frac{x}{2}=f(x) \Rightarrow f'(x)=\frac{1}{2}\sec^2\frac{x}{2}$

It is known that, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$

From equation (1), we obtain

$$\int\frac{e^x(1+\sin x)}{(1+\cos x)}dx=e^x\tan\frac{x}{2}+C$$

Question19. $e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$

Solution :

Let $I = \int e^x \left[\frac{1}{x} - \frac{1}{x^2}\right] dx$

Also, let $\frac{1}{x} = f(x) \Rightarrow f'(x) = \frac{-1}{x^2}$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

$\therefore I = \frac{e^x}{x} + C$

Question20. $\frac{(x-3)e^x}{(x-1)^3}$

Solution :

$$\int e^x \left\{\frac{x-3}{(x-1)^3}\right\} dx = \int e^x \left\{\frac{x-1-2}{(x-1)^3}\right\} dx$$
$$= \int e^x \left\{\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right\} dx$$

Let $f(x) = \frac{1}{(x-1)^2} \Rightarrow f'(x) = \frac{-2}{(x-1)^3}$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

$$\therefore \int e^x \left\{\frac{(x-3)}{(x-1)^2}\right\} dx = \frac{e^x}{(x-1)^2} + C$$

Question21. $e^{2x} \sin x$

Solution :

Let I = $\int e^{2x} \sin x$

Integrating by parts, we obtain

$$I = \sin x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \sin x \right) \int e^{2x} dx \right\} dx$$

$$\Rightarrow I = \sin x \cdot \frac{e^{2x}}{2} - \int \cos x \cdot \frac{e^{2x}}{2} dx$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \int e^{2x} \cos x \, dx$$

Again integrating by parts, we obtain

$$I = \frac{e^{2x} \cdot \sin x}{2} - \frac{1}{2} \left[\cos x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \cos x \right) \int e^{2x} dx \right\} dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \left[\cos x \cdot \frac{e^{2x}}{2} - \int (-\sin x) \frac{e^{2x}}{2} dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \cdot \sin x}{2} - \frac{1}{2} \left[\frac{e^{2x} \cos x}{2} + \frac{1}{2} \int e^{2x} \sin x dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{4} I \qquad \text{[From (1)]}$$

$$\Rightarrow I + \frac{1}{4} I = \frac{e^{2x} \cdot \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$\Rightarrow \frac{5}{4} I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$\Rightarrow I = \frac{4}{5} \left[\frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} \right] + C$$

$$\Rightarrow I = \frac{e^{2x}}{5} [2 \sin x - \cos x] + C$$

Question22. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Solution :

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$\Rightarrow \int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int 2\theta \cdot \sec^2\theta\, d\theta = 2\int \theta \cdot \sec^2\theta\, d\theta$$

Integrating by parts, we obtain

$$2\left[\theta \cdot \int \sec^2\theta d\theta - \int\left\{\left(\frac{d}{d\theta}\theta\right)\int \sec^2\theta d\theta\right\}d\theta\right]$$

$$= 2\left[\theta \cdot \tan\theta - \int \tan\theta d\theta\right]$$

$$= 2\left[\theta\tan\theta + \log|\cos\theta|\right] + C$$

$$= 2\left[x\tan^{-1}x + \log\left|\frac{1}{\sqrt{1+x^2}}\right|\right] + C$$

$$= 2x\tan^{-1}x + 2\log(1+x^2)^{-\frac{1}{2}} + C$$

$$= 2x\tan^{-1}x + 2\left[-\frac{1}{2}\log(1+x^2)\right] + C$$

$$= 2x\tan^{-1}x - \log(1+x^2) + C$$

Choose the correct answer in Exercise 23 and 24.

Question23. $\int x^2 e^{x^3} dx$ **equalsto**

(A) $\frac{1}{3}e^{x^3} + C$

(B) $\frac{1}{3}e^{x^2} + C$

(C) $\frac{1}{2}e^{x^3} + C$

(D) $\frac{1}{3}e^{x^2} + C$

Solution **:**

LetI= $\int x^2 e^{x^3} dx$

Also, let $x^3 = t \Rightarrow 3x^2 dx = dt$

$$\Rightarrow I = \frac{1}{3}\int e^t dt$$
$$= \frac{1}{3}\left(e^t\right) + C$$
$$= \frac{1}{3}e^{x^3} + C$$

Therefore,option,(A)iscorrect.

Question24. ∫equals: $e^x \sec x(1 + \tan x)dx$

(A) e^xcosx+C

(B) e^xsecx+C

(C) e^x sinx+C

(D) e^xtanx+C

Solution :

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$\Rightarrow \int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int 2\theta\cdot\sec^2\theta\, d\theta = 2\int \theta\cdot\sec^2\theta\, d\theta$$

Integrating by parts, we obtain

$$2\left[\theta\cdot\int\sec^2\theta d\theta - \int\left\{\left(\frac{d}{d\theta}\theta\right)\int\sec^2\theta d\theta\right\}d\theta\right]$$
$$= 2\left[\theta\cdot\tan\theta - \int\tan\theta d\theta\right]$$
$$= 2\left[\theta\tan\theta + \log|\cos\theta|\right] + C$$
$$= 2\left[x\tan^{-1}x + \log\left|\frac{1}{\sqrt{1+x^2}}\right|\right] + C$$
$$= 2x\tan^{-1}x + 2\log(1+x^2)^{-\frac{1}{2}} + C$$
$$= 2x\tan^{-1}x + 2\left[-\frac{1}{2}\log(1+x^2)\right] + C$$
$$= 2x\tan^{-1}x - \log(1+x^2) + C$$

Therefore, option (B) is correct.

Also, let $x^3 = t \Rightarrow 3x^2 dx = dt$

$$\Rightarrow I = \frac{1}{3}\int e^t dt$$
$$= \frac{1}{3}(e^t) + C$$
$$= \frac{1}{3}e^{x^3} + C$$

Let $I = \int e^x \sec x(1+\tan x)dx = \int e^x(\sec x + \sec x\tan x)dx$

Also, let $\sec x = f(x) \Rightarrow \sec x\tan x = f'(x)$

It is known that, $\int e^x\{f(x)+f'(x)\}dx = e^x f(x) + C$

$$\therefore I = e^x\sec x + C$$

Exercise 7.7 Page: 266

Question 1. $\sqrt{4-x^2}$

Solution :

$$\text{Let } I = \int\sqrt{4-x^2}\,dx = \int\sqrt{(2)^2-(x)^2}\,dx$$

$$\text{It is known that, } \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

$$\therefore I = \frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} + C$$

$$= \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + C$$

Question 2. $\sqrt{1-4x^2}$

Solution :

$$\text{Let } I = \int\sqrt{1-4x^2}\,dx = \int\sqrt{(1)^2-(2x)^2}\,dx$$

$$\text{Let } 2x = t \Rightarrow 2\,dx = dt$$

$$\therefore I = \frac{1}{2}\int\sqrt{(1)^2-(t)^2}\,dt$$

$$\text{It is known that, } \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

$$\Rightarrow I = \frac{1}{2}\left[\frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1}t\right] + C$$

$$= \frac{t}{4}\sqrt{1-t^2} + \frac{1}{4}\sin^{-1}t + C$$

$$= \frac{2x}{4}\sqrt{1-4x^2} + \frac{1}{4}\sin^{-1}2x + C$$

$$= \frac{x}{2}\sqrt{1-4x^2} + \frac{1}{4}\sin^{-1}2x + C$$

Question 3. $\sqrt{x^2+4x+6}$

Solution :

Let $I = \int \sqrt{x^2+4x+6}\ dx$

$= \int \sqrt{x^2+4x+4+2}\ dx$

$= \int \sqrt{\left(x^2+4x+4\right)+2}\ dx$

$= \int \sqrt{(x+2)^2+\left(\sqrt{2}\right)^2}\ dx$

It is known that, $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$\therefore I = \frac{(x+2)}{2}\sqrt{x^2+4x+6}+\frac{2}{2}\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$

$= \frac{(x+2)}{2}\sqrt{x^2+4x+6}+\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$

Question 4. $\sqrt{x^2+4x+1}$

Solution :

Let $I = \int \sqrt{x^2+4x+1}\,dx$

$= \int \sqrt{\left(x^2+4x+4\right)-3}\,dx$

$= \int \sqrt{(x+2)^2-\left(\sqrt{3}\right)^2}\ dx$

It is known that, $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$\therefore I = \frac{(x+2)}{2}\sqrt{x^2+4x+1}-\frac{3}{2}\log\left|(x+2)+\sqrt{x^2+4x+1}\right|+C$

Question 5. $\sqrt{1-4x-x^2}$

Solution :

Let $I = \int\sqrt{1-4x-x^2}\,dx$

$= \int\sqrt{1-\left(x^2+4x+4-4\right)}\,dx$

$= \int\sqrt{1+4-(x+2)^2}\,dx$

$= \int\sqrt{\left(\sqrt{5}\right)^2-(x+2)^2}\,dx$

It is known that, $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$\therefore I = \frac{(x+2)}{2}\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+C$

Question 6. $\sqrt{x^2+4x-5}$

Solution :

Let $I = \int\sqrt{x^2+4x-5}\,dx$

$= \int\sqrt{\left(x^2+4x+4\right)-9}\,dx$

$= \int\sqrt{(x+2)^2-(3)^2}\,dx$

It is known that, $\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$\therefore I = \frac{(x+2)}{2}\sqrt{x^2+4x-5}-\frac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right|+C$

Question 7. $\sqrt{1+3x-x^2}$

Solution :

$$\text{Let } I = \int\sqrt{1+3x-x^2}\,dx$$
$$= \int\sqrt{1-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)}\,dx$$
$$= \int\sqrt{\left(1+\frac{9}{4}\right)-\left(x-\frac{3}{2}\right)^2}\,dx$$
$$= \int\sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}\,dx$$

It is known that, $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$$\therefore I = \frac{x-\frac{3}{2}}{2}\sqrt{1+3x-x^2}+\frac{13}{4\times 2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right)+C$$
$$= \frac{2x-3}{4}\sqrt{1+3x-x^2}+\frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right)+C$$

Question 8. $\sqrt{x^2+3x}$

Solution :

$$\text{Let } I = \int\sqrt{x^2+3x}\,dx$$
$$= \int\sqrt{x^2+3x+\frac{9}{4}-\frac{9}{4}}\,dx$$
$$= \int\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}\,dx$$

It is known that, $\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$$\therefore I = \frac{\left(x+\frac{3}{2}\right)}{2}\sqrt{x^2+3x}-\frac{\frac{9}{4}}{2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right|+C$$
$$= \frac{(2x+3)}{4}\sqrt{x^2+3x}-\frac{9}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right|+C$$

Question 9. $\sqrt{1+\frac{x^2}{9}}$

Solution **:**

Let $I=\int\sqrt{1+\frac{x^2}{9}}dx=\frac{1}{3}\int\sqrt{9+x^2}\,dx=\frac{1}{3}\int\sqrt{(3)^2+x^2}\,dx$

It is known that, $\int\sqrt{x^2+a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$$\therefore I=\frac{1}{3}\left[\frac{x}{2}\sqrt{x^2+9}+\frac{9}{2}\log\left|x+\sqrt{x^2+9}\right|\right]+C$$
$$=\frac{x}{6}\sqrt{x^2+9}+\frac{3}{2}\log\left|x+\sqrt{x^2+9}\right|+C$$

Choose the correct answer in Exercise 10 to 11.

Question10. $\int\sqrt{1+x^2}dx$ **is equal to:**

(A) $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|\left(x+\sqrt{1+x^2}\right)+C\right|$

(B) $\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}+C$

(C) $\frac{2}{3}x\left(1+x^2\right)^{\frac{3}{2}}+C$

(D) $\frac{x^2}{2}\sqrt{1+x^2}+\frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right|+C$

Solution **:**

It is known that, $\int\sqrt{a^2+x^2}dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$\therefore\int\sqrt{1+x^2}dx=\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$

Therefore, option (A) is correct.

Question 11. $\int\sqrt{x^2-8x+7}dx$ **is equal to:**

(A) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+9\log\left|x-4+\ \sqrt{x^2-8x+7}\right|+C$
(B) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+9\log\left|x+4+\sqrt{x^2-8x+7}\right|+C$
(C) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$
(D) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|x-4+\ \sqrt{x^2-8x+7}\right|\ +C$

$$\begin{aligned}\text{Let } I &= \int\sqrt{x^2-8x+7}\,dx\\ &= \int\sqrt{\left(x^2-8x+16\right)-9}\,dx\\ &= \int\sqrt{(x-4)^2-(3)^2}\,dx\end{aligned}$$

It is known that, $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$$\therefore I=\frac{(x-4)}{2}\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|(x-4)+\sqrt{x^2-8x+7}\right|+C$$

Therefore, option (D) is correct.

Exercise 7.8 Page: 270

Question 1. $\int_a^b x\,dx$

Solution **:**
It is know that

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+...+f(a+(n-1)h)\right],\text{ where } h=\frac{b-a}{n}$$

Here, $a = a,\ b = b,$ and $f(x) = x$

$$\therefore \int_a^b x\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[a+(a+h)...(a+2h)...a+(n-1)h\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{a+a+a+...+a}_{n\text{ times}}\right)+\left(h+2h+3h+...+(n-1)h\right)\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left(1+2+3+...+(n-1)\right)\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+\frac{n(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{n}{n}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)(b-a)}{2n}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{\left(1-\frac{1}{n}\right)(b-a)}{2}\right]$$

$$=(b-a)\left[a+\frac{(b-a)}{2}\right]$$

$$=(b-a)\left[\frac{2a+b-a}{2}\right]$$

$$=\frac{(b-a)(b+a)}{2}$$

$$=\frac{1}{2}\left(b^2-a^2\right)$$

Question 2. $\int_0^5 (x+1)dx$

Solution :

Let I = $\int_0^5 (x+1)dx$

It is know that

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)...f(a+(n-1)h)\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=0$, $b=5$, and $f(x)=(x+1)$

$$\Rightarrow h=\frac{5-0}{n}=\frac{5}{n}$$

$$\therefore \int_0^5 (x+1)dx = (5-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{5}{n}\right)+...+f\left((n-1)\frac{5}{n}\right)\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{5}{n}+1\right)+...\left\{1+\left(\frac{5(n-1)}{n}\right)\right\}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1+1+1...1}_{n\text{ times}}\right)+\left[\frac{5}{n}+2\cdot\frac{5}{n}+3\cdot\frac{5}{n}+...(n-1)\frac{5}{n}\right]\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\{1+2+3...(n-1)\}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\cdot\frac{(n-1)n}{2}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5(n-1)}{2}\right]$$

$$=5\lim_{n\to\infty}\left[1+\frac{5}{2}\left(1-\frac{1}{n}\right)\right]$$

$$=5\left[1+\frac{5}{2}\right]$$

$$=5\left[\frac{7}{2}\right]$$

$$=\frac{35}{2}$$

Question 3. $\int_2^3 x^2dx$

Solution :

We know that

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+f(a+2h)\ldots f\{a+(n-1)h\}\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a = 2$, $b = 3$, and $f(x) = x^2$

$$\Rightarrow h = \frac{3-2}{n} = \frac{1}{n}$$

$$\therefore \int_2^3 x^2\,dx = (3-2)\lim_{n\to\infty}\frac{1}{n}\left[f(2)+f\left(2+\frac{1}{n}\right)+f\left(2+\frac{2}{n}\right)\ldots f\left\{2+(n-1)\frac{1}{n}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[(2)^2+\left(2+\frac{1}{n}\right)^2+\left(2+\frac{2}{n}\right)^2+\ldots\left(2+\frac{(n-1)}{n}\right)^2\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[2^2+\left\{2^2+\left(\frac{1}{n}\right)^2+2\cdot 2\cdot\frac{1}{n}\right\}+\ldots+\left\{(2)^2+\frac{(n-1)^2}{n^2}+2\cdot 2\cdot\frac{(n-1)}{n}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{2^2+\ldots+2^2}_{n \text{ times}}\right)+\left\{\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\ldots+\left(\frac{n-1}{n}\right)^2\right\}+2\cdot 2\cdot\left\{\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+\ldots+\frac{(n-1)}{n}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{1^2+2^2+3^2\ldots+(n-1)^2\right\}+\frac{4}{n}\left\{1+2+\ldots+(n-1)\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\}+\frac{4}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{n\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}+\frac{4n-4}{2}\right]$$

$$= \lim_{n\to\infty}\left[4+\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+2-\frac{2}{n}\right]$$

$$= 4+\frac{2}{6}+2$$

$$= \frac{19}{3}$$

Question 4. $\int_1^4 (x^2-x)\,dx$

Solution :

$$\text{Let } I=\int_1^4 \left(x^2-x\right)dx$$
$$=\int_1^4 x^2\,dx-\int_1^4 x\,dx$$
$$\text{Let } I=I_1-I_2,\ \text{where } I_1=\int_1^4 x^2dx \text{ and } I_2=\int_1^4 x\,dx \qquad ...(1)$$

We know that

$$\int_a^b f(x)\,dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+f\left(a+(n-1)h\right)\right],\ \text{where } h=\frac{b-a}{n}$$

For $I_1=\int_1^4 x^2dx,$

$a=1, b=4,$ and $f(x)=x^2$

$$\therefore h=\frac{4-1}{n}=\frac{3}{n}$$

$$I_1=\int_1^4 x^2dx=(4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1)+f(1+h)+...+f\left(1+(n-1)h\right)\right]$$
$$=3\lim_{n\to\infty}\frac{1}{n}\left[1^2+\left(1+\frac{3}{n}\right)^2+\left(1+2\cdot\frac{3}{n}\right)^2+...\left(1+\frac{(n-1)3}{n}\right)^2\right]$$
$$=3\lim_{n\to\infty}\frac{1}{n}\left[1^2+\left\{1^2+\left(\frac{3}{n}\right)^2+2\cdot\frac{3}{n}\right\}+...+\left\{1^2+\left(\frac{(n-1)3}{n}\right)^2+\frac{2\cdot(n-1)\cdot 3}{n}\right\}\right]$$
$$=3\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1^2+...+1^2}_{n\text{ times}}\right)+\left(\frac{3}{n}\right)^2\left\{1^2+2^2+...+(n-1)^2\right\}+2\cdot\frac{3}{n}\left\{1+2+...+(n-1)\right\}\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9}{n^2}\left\{\frac{(n-1)(n)(2n-1)}{6}\right\}+\frac{6}{n}\left\{\frac{(n-1)(n)}{2}\right\}\right]$$
$$=3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9n}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\frac{6n-6}{2}\right]$$
$$=3\lim_{n\to\infty}\left[1+\frac{9}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+3-\frac{3}{n}\right]$$
$$=3[1+3+3]$$
$$=3[7]$$
$$I_1=21 \qquad ...(2)$$

For $I_2=\int_1^4 x\,dx,$

$a=1, b=4,$ and $f(x)=x$

$$\Rightarrow h=\frac{4-1}{n}=\frac{3}{n}$$

$$\therefore I_2 = (4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1)+f(1+h)+\ldots f\left(a+(n-1)h\right)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1+(1+h)+\ldots+\left(1+(n-1)h\right)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1+\left(1+\frac{3}{n}\right)+\ldots+\left\{1+(n-1)\frac{3}{n}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1+1+\ldots+1}_{n\text{ times}}\right)+\frac{3}{n}\left(1+2+\ldots+(n-1)\right)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{3}{n}\left\{\frac{(n-1)n}{2}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{3}{2}\left(1-\frac{1}{n}\right)\right]$$

$$= 3\left[1+\frac{3}{2}\right]$$

$$= 3\left[\frac{5}{2}\right]$$

$$I_2 = \frac{15}{2} \qquad \ldots(3)$$

From equations (2) and (3), we obtain

$$I = I_1 + I_2 = 21 - \frac{15}{2} = \frac{27}{2}$$

Question 5. $\int_{-1}^{1} e^x dx$

Solution: LetI= $\int_{-1}^{1} e^x dx$

We know that

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)\ldots f(a+(n-1)h)\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=-1$, $b=1$, and $f(x)=e^x$

$$\therefore h=\frac{1+1}{n}=\frac{2}{n}$$

$$\therefore I=(1+1)\lim_{n\to\infty}\frac{1}{n}\left[f(-1)+f\left(-1+\frac{2}{n}\right)+f\left(-1+2\cdot\frac{2}{n}\right)+\ldots+f\left(-1+\frac{(n-1)2}{n}\right)\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}+e^{\left(-1+\frac{2}{n}\right)}+e^{\left(-1+2\cdot\frac{2}{n}\right)}+\ldots e^{\left(-1+(n-1)\frac{2}{n}\right)}\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}\left\{1+e^{\frac{2}{n}}+e^{\frac{4}{n}}+e^{\frac{6}{n}}+e^{(n-1)\frac{2}{n}}\right\}\right]$$

$$=2\lim_{n\to\infty}\frac{e^{-1}}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right]$$

$$=e^{-1}\times 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2-1}{e^{\frac{2}{n}}-1}\right]$$

$$=\frac{e^{-1}\times 2\left(e^2-1\right)}{\lim\limits_{\frac{2}{n}\to 0}\left(\frac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right)\times 2}$$

$$=e^{-1}\left[\frac{2\left(e^2-1\right)}{2}\right] \qquad \left[\lim_{h\to 0}\left(\frac{e^h-1}{h}\right)=1\right]$$

$$=\frac{e^2-1}{e}$$

$$=\left(e-\frac{1}{e}\right)$$

Question 6. $\int_0^4 \left(x+e^{2x}\right)dx$

Solution :

We know that

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=0$, $b=4$, and $f(x)=x+e^{2x}$

$$\therefore h=\frac{4-0}{n}=\frac{4}{n}$$

$$\Rightarrow \int_0^4\left(x+e^{2x}\right)dx=(4-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(h)+f(2h)+\ldots+f((n-1)h)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\left(0+e^0\right)+\left(h+e^{2h}\right)+\left(2h+e^{2\cdot 2h}\right)+\ldots+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[1+\left(h+e^{2h}\right)+\left(2h+e^{4h}\right)+\ldots+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\left\{h+2h+3h+\ldots+(n-1)h\right\}+\left(1+e^{2h}+e^{4h}+\ldots+e^{2(n-1)h}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[h\left\{1+2+\ldots(n-1)\right\}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\frac{\left(h(n-1)n\right)}{2}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\frac{4}{n}\cdot\frac{(n-1)n}{2}+\left(\frac{e^8-1}{e^{\frac{8}{n}}-1}\right)\right]$$

$$=4(2)+4\lim_{n\to\infty}\frac{\left(e^8-1\right)}{\left(\dfrac{e^{\frac{8}{n}}-1}{\frac{8}{n}}\right)8}$$

$$=8+\frac{4\cdot\left(e^8-1\right)}{8} \qquad \left(\lim_{x\to 0}\frac{e^x-1}{x}=1\right)$$

$$=8+\frac{e^8-1}{2}$$

$$=\frac{15+e^8}{2}$$

Exercise 7.9 Page: 273

Evaluate the definite integrals in Exercises 1 to 11.

Question1. $\int_{-1}^{1}(x+1)dx$

Solution :

Let $I=\int_{-1}^{1}(x+1)dx$

$$\int(x+1)\,dx=\frac{x^2}{2}+x=\mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$
\begin{aligned}
I &= F(1) - F(-1) \\
&= \left(\frac{1}{2}+1\right) - \left(\frac{1}{2}-1\right) \\
&= \frac{1}{2}+1-\frac{1}{2}+1 \\
&= 2
\end{aligned}
$$

Question2. $\int_2^3 \frac{1}{x} dx$

Solution :

Let $I = \int_2^3 \frac{1}{x}\, dx$

$\int \frac{1}{x} dx = \log|x| = F(x)$

By second fundamental theorem of calculus, we obtain

I = F(3) – F(2)
= log|3| – log|2| = log 3/2

Question3. $\int_1^2 (4x^3 - 5x^2 + 6x + 9)$

Solution :

Let $I = \int_1^2 (4x^3 - 5x^2 + 6x + 9) dx$

$$
\begin{aligned}
\int (4x^3 - 5x^2 + 6x + 9) dx &= 4\left(\frac{x^4}{4}\right) - 5\left(\frac{x^3}{3}\right) + 6\left(\frac{x^2}{2}\right) + 9(x) \\
&= x^4 - \frac{5x^3}{3} + 3x^2 + 9x = F(x)
\end{aligned}
$$

By second fundamental theorem of calculus, we obtain

$$I = F(2) - F(1)$$

$$I = \left\{2^4 - \frac{5\cdot(2)^3}{3} + 3(2)^2 + 9(2)\right\} - \left\{(1)^4 - \frac{5(1)^3}{3} + 3(1)^2 + 9(1)\right\}$$

$$= \left(16 - \frac{40}{3} + 12 + 18\right) - \left(1 - \frac{5}{3} + 3 + 9\right)$$

$$= 16 - \frac{40}{3} + 12 + 18 - 1 + \frac{5}{3} - 3 - 9$$

$$= 33 - \frac{35}{3}$$

$$= \frac{99 - 35}{3}$$

$$= \frac{64}{3}$$

Question4. $\int_0^{\frac{\pi}{4}} \sin 2x dx$

Solution **:**

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \sin 2x \, dx$$

$$\int \sin 2x \, dx = \left(\frac{-\cos 2x}{2}\right) = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{4}\right) - F(0)$$

$$= -\frac{1}{2}\left[\cos 2\left(\frac{\pi}{4}\right) - \cos 0\right]$$

$$= -\frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right]$$

$$= -\frac{1}{2}[0 - 1]$$

$$= \frac{1}{2}$$

Question5. $\int_0^{\frac{\pi}{2}} xos2xdx$

Solution **:**

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \cos 2x\, dx$$

$$\int \cos 2x\, dx = \left(\frac{\sin 2x}{2}\right) = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= \mathrm{F}\left(\frac{\pi}{2}\right) - \mathrm{F}(0) \\ &= \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0\right] \\ &= \frac{1}{2}[\sin\pi - \sin 0] \\ &= \frac{1}{2}[0-0] = 0 \end{aligned}$$

Question6. $\int_4^5 e^x dx$

Solution :

$$\text{Let } I = \int_4^5 e^x dx$$

$$\int e^x dx = e^x = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= \mathrm{F}(5) - \mathrm{F}(4) \\ &= e^5 - e^4 \\ &= e^4(e-1) \end{aligned}$$

Question7. $\int_0^{\frac{\pi}{4}} \tan x dx$

Solution **:**

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \tan x\, dx$$

$$\int \tan x\, dx = -\log|\cos x| = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{4}\right) - F(0)$$
$$= -\log\left|\cos\frac{\pi}{4}\right| + \log|\cos 0|$$
$$= -\log\left|\frac{1}{\sqrt{2}}\right| + \log|1|$$
$$= -\log(2)^{-\frac{1}{2}}$$
$$= \frac{1}{2}\log 2$$

Question8. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos ec x dx$

Solution :

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos ec\, x\, dx$

$$\int \operatorname{cosec} x \; dx = \log|\operatorname{cosec} x - \cot x| = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{4}\right) - F\left(\frac{\pi}{6}\right)$$
$$= \log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right| - \log\left|\operatorname{cosec}\frac{\pi}{6} - \cot\frac{\pi}{6}\right|$$
$$= \log\left|\sqrt{2} - 1\right| - \log\left|2 - \sqrt{3}\right|$$
$$= \log\left(\frac{\sqrt{2} - 1}{2 - \sqrt{3}}\right)$$

Question9. $\int_0^4 \frac{dx}{\sqrt{1 - x^2}}$

Solution :

Let $I = \int_0^1 \frac{dx}{\sqrt{1 - x^2}}$

$$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(1) - F(0)$$
$$= \sin^{-1}(1) - \sin^{-1}(0)$$
$$= \frac{\pi}{2} - 0$$
$$= \frac{\pi}{2}$$

Question10. $\int_0^4 \frac{dx}{1+x^2}$

Solution :

$$\text{Let } I = \int_0^4 \frac{dx}{1+x^2}$$
$$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(1) - F(0)$$
$$= \tan^{-1}(1) - \tan^{-1}(0)$$
$$= \frac{\pi}{4}$$

Question11. $\int_2^3 \frac{dx}{x^2-1}$

Solution :

$$\text{Let } I = \int_2^3 \frac{dx}{x^2-1}$$
$$\int \frac{dx}{x^2-1} = \frac{1}{2} \log\left|\frac{x-1}{x+1}\right| = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(3) - F(2)$$
$$= \frac{1}{2}\left[\log\left|\frac{3-1}{3+1}\right| - \log\left|\frac{2-1}{2+1}\right|\right]$$
$$= \frac{1}{2}\left[\log\left|\frac{2}{4}\right| - \log\left|\frac{1}{3}\right|\right]$$
$$= \frac{1}{2}\left[\log\frac{1}{2} - \log\frac{1}{3}\right]$$
$$= \frac{1}{2}\left[\log\frac{3}{2}\right]$$

Evaluate the definite integrals in Exercises 12 to 20.

Question12. $\int_0^{\frac{\pi}{2}} \cos^2 x dx$

Solution :

Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

$$\int \cos^2 x \, dx = \int \left(\frac{1+\cos 2x}{2}\right) dx = \frac{x}{2} + \frac{\sin 2x}{4} = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \left[F\left(\frac{\pi}{2}\right) - F(0)\right]$$
$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\sin \pi}{2}\right) - \left(0 + \frac{\sin 0}{2}\right)\right]$$
$$= \frac{1}{2}\left[\frac{\pi}{2} + 0 - 0 - 0\right]$$
$$= \frac{\pi}{4}$$

Question13. $\int_2^3 \frac{x dx}{x^2 + 1}$

Solution :

Let $I = \int_2^3 \frac{x}{x^2+1} dx$

$$\int \frac{x}{x^2+1} dx = \frac{1}{2}\int \frac{2x}{x^2+1} dx = \frac{1}{2}\log\left(1+x^2\right) = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(3) - F(2)$$
$$= \frac{1}{2}\left[\log\left(1+(3)^2\right) - \log\left(1+(2)^2\right)\right]$$
$$= \frac{1}{2}\left[\log(10) - \log(5)\right]$$
$$= \frac{1}{2}\log\left(\frac{10}{5}\right) = \frac{1}{2}\log 2$$

Question14. $\int_0^1 \frac{2x+3}{5x^2+1}dx$

Solution :

$$\text{Let } I = \int_0^1 \frac{2x+3}{5x^2+1}dx$$
$$\int \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\int \frac{5(2x+3)}{5x^2+1}dx$$
$$= \frac{1}{5}\int \frac{10x+15}{5x^2+1}dx$$
$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5x^2+1}dx$$
$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5\left(x^2+\frac{1}{5}\right)}dx$$
$$= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{5}\cdot\frac{1}{\frac{1}{\sqrt{5}}}\tan^{-1}\frac{x}{\frac{1}{\sqrt{5}}}$$
$$= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}x\right)$$
$$= F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(1) - F(0)$$
$$= \left\{\frac{1}{5}\log(5+1) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\right)\right\} - \left\{\frac{1}{5}\log(1) + \frac{3}{\sqrt{5}}\tan^{-1}(0)\right\}$$
$$= \frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$$

Question15. $\int_0^1 xe^{x^2}dx$

Solution :

Let $I = \int_0^1 x e^{x^2} dx$

Put $x^2 = t \Rightarrow 2x\, dx = dt$

As $x \to 0, t \to 0$ and as $x \to 1, t \to 1$,

$\therefore I = \frac{1}{2}\int_0^1 e^t dt$

$\frac{1}{2}\int e^t dt = \frac{1}{2}e^t = \text{F}(t)$

By second fundamental theorem of calculus, we obtain

$$I = \text{F}(1) - \text{F}(0)$$
$$= \frac{1}{2}e - \frac{1}{2}e^0$$
$$= \frac{1}{2}(e-1)$$

Question16. $\int_1^2 \frac{5x^2}{x^2+4x+3}$

Solution :

Let $I = \int_1^2 \frac{5x^2}{x^2+4x+3} dx$

Dividing $5x^2$ by x^2+4x+3, we obtain

$$I = \int_1^2 \left\{5 - \frac{20x+15}{x^2+4x+3}\right\} dx$$
$$= \int_1^2 5dx - \int_1^2 \frac{20x+15}{x^2+4x+3} dx$$
$$= [5x]_1^2 - \int_1^2 \frac{20x+15}{x^2+4x+3} dx$$
$$I = 5 - I_1, \text{ where } I = \int_1^2 \frac{20x+15}{x^2+4x+3} dx \qquad \ldots(1)$$

Consider $I_1 = \int_1^2 \frac{20x+15}{x^2+4x+8} dx$

$$\text{Let } 20x+15 = \text{A}\frac{d}{dx}(x^2+4x+3)+\text{B}$$
$$= 2\text{A}x + (4\text{A}+\text{B})$$

Equating the coefficients of x and constant term, we obtain

A = 10 and B = −25

$$\Rightarrow I_1 = 10\int_1^2 \frac{2x+4}{x^2+4x+3}dx - 25\int_1^2 \frac{dx}{x^2+4x+3}$$

Let $x^2+4x+3=t$

$$\Rightarrow (2x+4)dx = dt$$

$$\Rightarrow I_1 = 10\int \frac{dt}{t} - 25\int \frac{dx}{(x+2)^2-1^2}$$

$$= 10\log t - 25\left[\frac{1}{2}\log\left(\frac{x+2-1}{x+2+1}\right)\right]$$

$$= \left[10\log\left(x^2+4x+3\right)\right]_1^2 - 25\left[\frac{1}{2}\log\left(\frac{x+1}{x+3}\right)\right]_1^2$$

$$= [10\log 15 - 10\log 8] - 25\left[\frac{1}{2}\log\frac{3}{5} - \frac{1}{2}\log\frac{2}{4}\right]$$

$$= \left[10\log(5\times3) - 10\log(4\times2)\right] - \frac{25}{2}[\log 3 - \log 5 - \log 2 + \log 4]$$

$$= [10\log 5 + 10\log 3 - 10\log 4 - 10\log 2] - \frac{25}{2}[\log 3 - \log 5 - \log 2 + \log 4]$$

$$= \left[10+\frac{25}{2}\right]\log 5 + \left[-10-\frac{25}{2}\right]\log 4 + \left[10-\frac{25}{2}\right]\log 3 + \left[-10+\frac{25}{2}\right]\log 2$$

$$= \frac{45}{2}\log 5 - \frac{45}{2}\log 4 - \frac{5}{2}\log 3 + \frac{5}{2}\log 2$$

$$= \frac{45}{2}\log\frac{5}{4} - \frac{5}{2}\log\frac{3}{2}$$

Substituting the value of I_1 in (1), we obtain

$$I = 5 - \left[\frac{45}{2}\log\frac{5}{4} - \frac{5}{2}\log\frac{3}{2}\right]$$

$$= 5 - \frac{5}{2}\left[9\log\frac{5}{4} - \log\frac{3}{2}\right]$$

Question17. $\int_0^{\frac{\pi}{4}} (2\sec^2 x + x^3 + 2)dx$

Solution :

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \left(2\sec^2 x + x^3 + 2\right) dx$$

$$\int \left(2\sec^2 x + x^3 + 2\right) dx = 2\tan x + \frac{x^4}{4} + 2x = \text{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \text{F}\left(\frac{\pi}{4}\right) - \text{F}(0)$$

$$= \left\{\left(2\tan\frac{\pi}{4} + \frac{1}{4}\left(\frac{\pi}{4}\right)^4 + 2\left(\frac{\pi}{4}\right)\right) - \left(2\tan 0 + 0 + 0\right)\right\}$$

$$= 2\tan\frac{\pi}{4} + \frac{\pi^4}{4^5} + \frac{\pi}{2}$$

$$= 2 + \frac{\pi}{2} + \frac{\pi^4}{1024}$$

Question18. $\int_0^{\Pi} \left(\frac{\sin^2 x}{2} - \frac{\cos^2 x}{2}\right) dx$

Solution :

$$\text{Let } I = \int_0^{\pi} \left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right) dx$$

$$= -\int_0^{\pi} \left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) dx$$

$$= -\int_0^{\pi} \cos x \, dx$$

$$\int \cos x \, dx = \sin x = \text{F}(x)$$

By second fundamental theorem of calculus, we obtain

I = F(π) – F(0)

= sin π – sin 0

= 0

Question19. $\int_0^{2} \frac{6x+3}{x^2+4}$ dx

Solution **:**

$$\text{Let } I = \int_0^2 \frac{6x+3}{x^2+4}dx$$

$$\int \frac{6x+3}{x^2+4}dx = 3\int \frac{2x+1}{x^2+4}dx$$

$$= 3\int \frac{2x}{x^2+4}dx + 3\int \frac{1}{x^2+4}dx$$

$$= 3\log\left(x^2+4\right) + \frac{3}{2}\tan^{-1}\frac{x}{2} = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(2) - \mathrm{F}(0)$$

$$= \left\{3\log\left(2^2+4\right) + \frac{3}{2}\tan^{-1}\left(\frac{2}{2}\right)\right\} - \left\{3\log(0+4) + \frac{3}{2}\tan^{-1}\left(\frac{0}{2}\right)\right\}$$

$$= 3\log 8 + \frac{3}{2}\tan^{-1}1 - 3\log 4 - \frac{3}{2}\tan^{-1}0$$

$$= 3\log 8 + \frac{3}{2}\left(\frac{\pi}{4}\right) - 3\log 4 - 0$$

$$= 3\log\left(\frac{8}{4}\right) + \frac{3\pi}{8}$$

$$= 3\log 2 + \frac{3\pi}{8}$$

Question20. $\int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)$

Solution **:**

$$\text{Let } I = \int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)dx$$

$$\int \left(xe^x + \sin\frac{\pi x}{4}\right)dx = x\int e^x dx - \int\left\{\left(\frac{d}{dx}x\right)\int e^x dx\right\}dx + \left\{\frac{-\cos\frac{\pi x}{4}}{\frac{\pi}{4}}\right\}$$

$$= xe^x - \int e^x dx - \frac{4\pi}{\pi}\cos\frac{x}{4}$$

$$= xe^x - e^x - \frac{4\pi}{\pi}\cos\frac{x}{4}$$

$$= \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(1) - F(0)$$
$$= \left(1.e^1 - e^1 - \frac{4}{\pi}\cos\frac{\pi}{4}\right) - \left(0.e^0 - e^0 - \frac{4}{\pi}\cos 0\right)$$
$$= e - e - \frac{4}{\pi}\left(\frac{1}{\sqrt{2}}\right) + 1 + \frac{4}{\pi}$$
$$= 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$$

Choose the correct answer in Exercises 21 and 22.

Question21. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ **equals:**

(A)π/3
(B)2π/3
(C)π/6
(D)π/12

Solution :

$$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = F(\sqrt{3}) - F(1)$$
$$= \tan^{-1}\sqrt{3} - \tan^{-1} 1$$
$$= \frac{\pi}{3} - \frac{\pi}{4}$$
$$= \frac{\pi}{12}$$

Therefore, option (D) is correct.

Question22. $\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$ **equals:**

(A)π/6
(B)π/12
(C)π/24
(D)π/4

Solution :

$$\int \frac{dx}{4+9x^2} = \int \frac{dx}{(2)^2+(3x)^2}$$

Put $3x = t \Rightarrow 3dx = dt$

$$\therefore \int \frac{dx}{(2)^2+(3x)^2} = \frac{1}{3}\int \frac{dt}{(2)^2+t^2}$$

$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)$$

$$= F(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2} = F\left(\frac{2}{3}\right) - F(0)$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3}{2}\cdot\frac{2}{3}\right) - \frac{1}{6}\tan^{-1}0$$

$$= \frac{1}{6}\tan^{-1}1 - 0$$

$$= \frac{1}{6}\times\frac{\pi}{4}$$

$$= \frac{\pi}{24}$$

Therefore, option (C) is correct.

Exercise 7.10 Page: 280

Question:1 By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx$$

Answer:

We have $I = \int_0^{\frac{\pi}{2}} \cos^2 x dx$ ………………………………………………… (i)

By using

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

We get :-

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x dx = \int_0^{\frac{\pi}{2}} \cos^2 (\frac{\pi}{2} - x)dx$$

or

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x dx$$ .... (ii)

Adding both (i) and (ii), we get :-

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx + \int_0^{\frac{\pi}{2}} \sin^2 x dx = 2I$$

or $\int_0^{\frac{\pi}{2}} (cos^2 x + sin^2 x)dx = 2I$

or $\int_0^{\frac{\pi}{2}} 1.dx = 2I$

or $2I = [x]_0^{\frac{\Pi}{2}} = \frac{\Pi}{2}$

or $I = \frac{\Pi}{4}$

Question:2 By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

Answer:

We have $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ .... **(i)**

By using ,

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

We get,

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin(\frac{\pi}{2} - x)}}{\sqrt{\sin(\frac{\pi}{2} - x)} + \sqrt{\cos(\frac{\pi}{2} - x)}} dx$$

or $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$ **..(ii)**

Adding (i) and (ii), we get,

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

or $2I = \int_0^{\frac{\pi}{2}} 1.dx$

or $2I = [x]_0^{\frac{\Pi}{2}} = \frac{\Pi}{2}$

Thus $I = \frac{\Pi}{4}$

Question:3 By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$$

Answer:

We have $I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$ ....(i)

By using :

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

We get,

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}}(\frac{\pi}{2} - x)dx}{\sin^{\frac{3}{2}}(\frac{\pi}{2} - x) + \cos^{\frac{3}{2}}(\frac{\pi}{2} - x)}$$

or $I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}} x dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$ . ...(ii)

Adding (i) and (ii), we get :

$$2I = \int_0^{\frac{\pi}{2}} \frac{(sin^{\frac{3}{2}}x + cos^{\frac{3}{2}}x)dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$$

or $2I = \int_0^{\frac{\pi}{2}} 1.dx$

or $2I = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

Thus $I = \frac{\pi}{4}$

Question:4 By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

$\int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x}$.

Answer:

We have $I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x}$ ....(i)

By using :

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

We get,

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^5(\frac{\pi}{2}-x)dx}{\sin^5(\frac{\pi}{2}-x)+\cos^5(\frac{\pi}{2}-x)}$$

or $I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x dx}{\sin^5 x + \cos^5 x}$ . ...(ii)

Adding (i) and (ii), we get :

$$2I = \int_0^{\frac{\pi}{2}} \frac{(sin^5 x + cos^5 x)dx}{\sin^5 x + \cos^5 x}$$

or $2I = \int_0^{\frac{\pi}{2}} 1.dx$

or $2I = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

Thus $I = \frac{\pi}{4}$

Question:5 **By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.**

$$\int_{-5}^{5} |x+2|dx$$

Answer:

We have, $I = \int_{-5}^{5} |x+2|dx$

For opening the modulas we need to define the bracket :

If (x + 2) < 0 then x belongs to (-5, -2). And if (x + 2) > 0 then x belongs to (-2, 5).

So the integral becomes :-

$$I = \int_{-5}^{-2} -(x+2)dx + \int_{-2}^{5}(x+2)dx$$

or $I = -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$

This gives $I = 29$

Question:6 By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

$$\int_{2}^{8} |x-5|dx$$

Answer:

We have, $I = \int_{2}^{8} |x-5|dx$

For opening the modulas we need to define the bracket :

If (x - 5) < 0 then x belongs to (2, 5). And if (x - 5) > 0 then x belongs to (5, 8).

So the integral becomes:-

$$I = \int_{2}^{5} -(x-5)dx + \int_{5}^{8}(x-5)dx$$

or $I = -\left[\frac{x^2}{2} - 5x\right]_{2}^{5} + \left[\frac{x^2}{2} - 5x\right]_{5}^{8}$

This gives $I = 9$

Question:7 **By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.**

$$\int_0^1 x(1-x)^n dx$$

Answer:

We have $I = \int_0^1 x(1-x)^n dx$

U sing the property : -

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

We get : -

$$I = \int_0^1 x(1-x)^n dx = \int_0^1 (1-x)(1-(1-x))^n dx$$

or $I = \int_0^1 (1-x)x^n \, dx$

or $I = \int_0^1 (x^n - x^{n+1}) \, dx$

or $= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$

or $= \left[\frac{1}{n+1} - \frac{1}{n+2}\right]$

or $I = \frac{1}{(n+1)(n+2)}$

Question:8 **By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.**

$$\int_0^{\frac{\pi}{4}} \log(1+\tan x)dx$$

Answer:

We have $I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)dx$

By using the identity

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

We get,

$$I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)dx = \int_0^{\frac{\pi}{4}} \log(1+\tan(\frac{\pi}{4}-x))dx$$

or $I = \int_0^{\frac{\pi}{4}} \log(1+\frac{1-\tan x}{1+\tan x})dx$

or $I = \int_0^{\frac{\pi}{4}} \log(\frac{2}{1+\tan x})dx$

or $I = \int_0^{\frac{\pi}{4}} \log 2dx - \int_0^{\frac{\pi}{4}} \log(1+\tan x)dx$

or $I = \int_0^{\frac{\pi}{4}} \log 2dx - I$

or $2I = [x\log 2]_0^{\frac{\Pi}{4}}$

or $I = \frac{\Pi}{8}\log 2$

Question:9 **By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.**

$$\int_0^2 x\sqrt{2-x}dx$$

Answer:

We have $I = \int_0^2 x\sqrt{2-x}dx$

By using the identity

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

We get :

$$I = \int_0^2 x\sqrt{2-x}dx = \int_0^2 (2-x)\sqrt{2-(2-x)}dx$$

or $I = \int_0^2 (2-x)\sqrt{x}dx$

or $I = \int_0^2 (2\sqrt{x} - x^{\frac{3}{2}}dx$

or $= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^2$

or $= \frac{4}{3}(2)^{\frac{3}{2}} - \frac{2}{5}(2)^{\frac{5}{2}}$

or $I = \frac{16\sqrt{2}}{15}$

Question:10 **By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.**

$$\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)dx$$

Answer:

We have $I = \int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)dx$

or $I = \int_0^{\frac{\pi}{2}} (2\log\sin x - \log(2\sin x\cos x))dx$

or $I = \int_0^{\frac{\pi}{2}} (\log \sin x - \log \cos x - \log 2) dx$ ....(i)

By using the identity :

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx$$

We get :

$$I = \int_0^{\frac{\pi}{2}} (\log \sin(\frac{\pi}{2} - x) - \log \cos(\frac{\pi}{2} - x) - \log 2) dx$$

or $I = \int_0^{\frac{\pi}{2}} (\log \cos x - \log \sin x - \log 2) dx$ ....(ii)

Adding (i) and (ii) we get :-

$$2I = \int_0^{\frac{\pi}{2}} (-\log 2 - \log 2) dx$$

or $I = -\log 2 \left[\frac{\Pi}{2} \right]$

or $I = \frac{\Pi}{2} \log \frac{1}{2}$

Question:11 By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x dx$$

Answer:

We have $I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x dx$

We know that $\sin^2 x$ is an even function. i.e., $\sin^2(-x) = (-\sin x)^2 = \sin^2 x$.

Also,

$$I = \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

So,

$$I = 2\int_{0}^{\frac{\pi}{2}} \sin^2 x dx = 2\int_{0}^{\frac{\pi}{2}} \frac{(1-\cos 2x)}{2}dx$$

or $= \left[x - \frac{\sin 2x}{2}\right]_{0}^{\frac{\Pi}{2}}$

or $I = \frac{\Pi}{2}$

Question:12 By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

$$\int_{0}^{\pi} \frac{xdx}{1+\sin x}$$

Answer:

We have $I = \int_{0}^{\pi} \frac{xdx}{1+\sin x}$ **..(i)**

By using the identity :-

$$\int_{0}^{a} f(x)dx = \int_{0}^{a} f(a-x)dx$$

We get,

$$I = \int_{0}^{\pi} \frac{xdx}{1+\sin x} = \int_{0}^{\pi} \frac{(\Pi - x)dx}{1+\sin(\Pi - x)}$$

or $I = \int_{0}^{\pi} \frac{(\Pi - x)dx}{1+\sin x}$ **..(ii)**

Adding both (i) and (ii) we get,

$$2I = \int_0^{\pi} \frac{\Pi}{1+\sin x} dx$$

or $$2I = \Pi \int_0^{\pi} \frac{1-\sin x}{(1+\sin x)(1-\sin x)} dx = \Pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx$$

or $$2I = \Pi \int_0^{\pi} (\sec^2 - \tan x \sec x) x dx$$

or $I = \Pi$

Question:13 By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$$

Answer:

We have $$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$$

We know that $\sin^7 x$ is an odd function.

So the following property holds here:-

$$\int_{-a}^{a} f(x) dx = 0$$

Hence

$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx = 0$$

Question:14 **By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.**

$$\int_0^{2\pi} \cos^5 x dx$$

Answer:

We have $I = \int_0^{2\pi} \cos^5 x dx$

I t is known that :-

$\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$ If f (2a - x) = f(x)

$= 0$ If f (2a - x) = - f(x)

Now, using the above property

$$\cos^5(\Pi - x) = -\cos^5 x$$

Therefore, $I = 0$

Question:15 **By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.**

$$\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$$

Answer:

We have $I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$ **..(i)**

By using the property :-

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

We get ,

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin(\frac{\pi}{2}-x)-\cos(\frac{\pi}{2}-x)}{1+\sin(\frac{\pi}{2}-x)\cos(\frac{\pi}{2}-x)}dx$$

or $I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin x \cos x}dx$ **..(ii)**

Adding both (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{0}{1+\sin x \cos x}dx$$

Thus I = 0

Question:16 By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

$$\int_0^{\pi} \log(1+\cos x)dx$$

Answer:

We have $I = \int_0^{\pi} \log(1+\tan x)dx$ ....
.....(i)

By using the property:-

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

We get,

or

$$I = \int_0^{\pi} \log(1 + \cos(\Pi - x))dx$$

$$I = \int_0^{\pi} \log(1 - \cos x)dx$$ ....(ii)

Adding both (i) and (ii) we get,

$$2I = \int_0^{\pi} \log(1 + \cos x)dx + \int_0^{\pi} \log(1 - \cos x)dx$$

or $$2I = \int_0^{\pi} \log(1 - \cos^2 x)dx = \int_0^{\pi} \log \sin^2 x dx$$

or $$2I = 2\int_0^{\pi} \log \sin x dx$$

or $$I = \int_0^{\pi} \log \sin x dx$$ ....(iii)

or $$I = 2\int_0^{\frac{\pi}{2}} \log \sin x dx$$ ....(iv)

or $$I = 2\int_0^{\frac{\pi}{2}} \log \cos x dx$$ ....(v)

Adding (iv) and (v) we get,

$$I = -\pi \log 2$$

Question:17 **By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.**

$$\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$$

Answer:

We have $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$ **..(i)**

By using, we get

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

We get,

$I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx = \int_0^a \frac{\sqrt{(a-x)}}{\sqrt{(a-x)}+\sqrt{x}}dx$ **..(ii)**

Adding (i) and (ii) we get :

$$2I = \int_0^a \frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx$$

or $2I = [x]_0^a = a$

or $I = \frac{a}{2}$

Question:18 **By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.**

$$\int_0^4 |x-1|dx$$

Answer:

We have, $I = \int_0^4 |x-1|dx$

For opening the modulas we need to define the bracket :

If (x - 1) < 0 then x belongs to (0, 1). And if (x - 1) > 0 then x belongs to (1, 4).

So the integral becomes:-

$$I = \int_0^1 -(x-1)dx + \int_1^4 (x-1)dx$$

or $I = \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^4$

This gives $I = 5$

Question:19 Show that $\int_0^a f(x)g(x)dx = 2\int_0^a f(x)dx$ **if** f **and** g **are defined as** $f(x) = f(a-x)$ **and** $g(x) + g(a-x) = 4$

Answer:

Let $I = \int_0^a f(x)g(x)dx$ ....(i)

This can also be written as :

$$I = \int_0^a f(a-x)g(a-x)dx$$

or $I = \int_0^a f(x)g(a-x)dx$ ....(ii)

Adding (i) and (ii), we get,

$$2I = \int_0^a f(x)g(a-x)dx + \int_0^a f(x)g(x)dx$$

$$2I = \int_0^a f(x)4dx$$

or $I = 2\int_0^a f(x)dx$

Question:20 **Choose the correct answer in Exercises 20 and 21.**

The value of is $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)dx$ is

(A) 0

(B) 2

(C) π

(D) 1

Answer:

We have

$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)dx$$

This can be written as :

$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x^3 dx + \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x\cos x + \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \tan^5 x + \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 1dx$$

Also if a function is even function then $\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx$

And if the function is an odd function then : $\int_{-a}^{a} f(x)\ dx = 0$

Using the above property I become:-

$$I \ = \ 0+0+0+2\int_0^{\frac{\Pi}{2}} 1.dx$$

or $I \ = \ 2\left[x\right]_0^{\frac{\Pi}{2}}$

or $I \ = \ \Pi$

Question:21 Choose the correct answer in Exercises 20 and 21.

The value of $\int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ is

Answer:

We have

$I \ = \ \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ ....(i)

By using :

$$\int_0^a f(x)dx \ = \ \int_0^a f(a-x)dx$$

We get,

$$I \ = \ \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx \ = \ \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin(\frac{\pi}{2}-x)}{4+3\cos(\frac{\pi}{2}-x)}\right)dx$$

or $I \ = \ \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx$ ....(ii)

Adding (i) and (ii), we get:

$$2I \ = \ \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx \ + \ \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx$$

or $2I = \int_0^{\frac{\pi}{2}} \log 1.dx$

Thus $I = 0$

Chapter 7 Miscellaneous Exercise

NCERT Solutions for Class 12 Maths Chapter 7 Miscellaneous Exercise
Question 1:

$\frac{1}{x-x^3}$

Solution:

$$\frac{1}{x-x^3}=\frac{1}{x\left(1-x^2\right)}=\frac{1}{x(1-x)(1+x)}$$

$$\text{Let } \frac{1}{x(1-x)(1+x)}=\frac{A}{x}+\frac{B}{(1-x)}+\frac{C}{1+x} \qquad \text{...(1)}$$

$$\Rightarrow 1=A\left(1-x^2\right)+Bx(1+x)+Cx(1-x)$$

$$\Rightarrow 1=A-Ax^2+Bx+Bx^2+Cx-Cx^2$$

Equating the coefficients of x^2, x, and constant term, we obtain

$-A+B-C=0$

$B+C=0$

$A=1$

On solving these equations, we obtain

$$A=1, B=\frac{1}{2}, \text{ and } C=-\frac{1}{2}$$

From equation (1), we obtain

$$\frac{1}{x(1-x)(1+x)}=\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}$$

$$\Rightarrow \int\frac{1}{x(1-x)(1+x)}dx=\int\frac{1}{x}dx+\frac{1}{2}\int\frac{1}{1-x}dx-\frac{1}{2}\int\frac{1}{1+x}dx$$

$$=\log|x|-\frac{1}{2}\log|(1-x)|-\frac{1}{2}\log|(1+x)|$$

$$=\log|x|-\log\left|(1-x)^{\frac{1}{2}}\right|-\log\left|(1+x)^{\frac{1}{2}}\right|$$

$$=\log\left|\frac{x}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}}\right|+C$$

$$=\log\left|\left(\frac{x^2}{1-x^2}\right)^{\frac{1}{2}}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right|+C$$

Question 2:

$$\frac{1}{\sqrt{x+a}+\sqrt{(x+b)}}$$

Solution:

$$\frac{1}{\sqrt{x+a}+\sqrt{x+b}}=\frac{1}{\sqrt{x+a}+\sqrt{x+b}}\times\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$=\frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}$$

$$=\frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{a-b}$$

$$\Rightarrow \int\frac{1}{\sqrt{x+a}-\sqrt{x+b}}dx=\frac{1}{a-b}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)dx$$

$$=\frac{1}{(a-b)}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right]$$

$$=\frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right]+\mathrm{C}$$

Question 3:

$\frac{1}{x\sqrt{ax-x^2}}$ **[Hint: Put** $x=\frac{a}{t}$ **]**

Solution:

$$\frac{1}{x\sqrt{ax-x^2}}$$

Let $x=\frac{a}{t} \Rightarrow \; dx=-\frac{a}{t^2}dt$

$$\Rightarrow \int \frac{1}{x\sqrt{ax-x^2}}dx = \int \frac{1}{\frac{a}{t}\sqrt{a\cdot\frac{a}{t}-\left(\frac{a}{t}\right)^2}}\left(-\frac{a}{t^2}dt\right)$$

$$= -\int \frac{1}{at}\cdot\frac{1}{\sqrt{\frac{1}{t}-\frac{1}{t^2}}}dt$$

$$= -\frac{1}{a}\int \frac{1}{\sqrt{\frac{t^2}{t}-\frac{t^2}{t^2}}}dt$$

$$= -\frac{1}{a}\int \frac{1}{\sqrt{t-1}}dt$$

$$= -\frac{1}{a}\left[2\sqrt{t-1}\right]+\text{C}$$

$$= -\frac{1}{a}\left[2\sqrt{\frac{a}{x}-1}\right]+\text{C}$$

$$= -\frac{2}{a}\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right)+\text{C}$$

$$= -\frac{2}{a}\left(\sqrt{\frac{a-x}{x}}\right)+\text{C}$$

Question 4:

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Solution:

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Multiplying and dividing by x^{-3}, we obtain

$$\frac{x^{-3}}{x^2\cdot x^{-3}\left(x^4+1\right)^{\frac{3}{4}}}=\frac{x^{-3}\left(x^4+1\right)^{\frac{-3}{4}}}{x^2\cdot x^{-3}}$$

$$=\frac{\left(x^4+1\right)^{\frac{-3}{4}}}{x^5\cdot\left(x^4\right)^{-\frac{3}{4}}}$$

$$=\frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{-\frac{3}{4}}$$

$$=\frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}$$

Let $\frac{1}{x^4}=t \Rightarrow -\frac{4}{x^5}dx=dt \Rightarrow \frac{1}{x^5}dx=-\frac{dt}{4}$

$$\therefore \int\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}dx=\int\frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}dx$$

$$=-\frac{1}{4}\int(1+t)^{-\frac{3}{4}}dt$$

$$=-\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right]+C$$

$$=-\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}{\frac{1}{4}}+C$$

$$=-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C$$

Question 5:

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}\quad\left[\text{Hint: }\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}\text{ Put } x=t^6\right]$$

Solution:

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$$

Let $x=t^6 \Rightarrow dx=6t^5dt$

$$\therefore \int\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx=\int\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}dx$$

$$=\int\frac{6t^5}{t^2(1+t)}dt$$

$$=6\int\frac{t^3}{(1+t)}dt$$

On dividing, we obtain

$$\int\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx=6\int\left\{\left(t^2-t+1\right)-\frac{1}{1+t}\right\}dt$$

$$=6\left[\left(\frac{t^3}{3}\right)-\left(\frac{t^2}{2}\right)+t-\log|1+t|\right]$$

$$=2x^{\frac{1}{2}}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1+x^{\frac{1}{6}}\right)+C$$

$$=2\sqrt{x}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1+x^{\frac{1}{6}}\right)+C$$

Question 6:

$$\frac{5x}{(x+1)\left(x^2+9\right)}$$

Solution:

Let $\dfrac{5x}{(x+1)(x^2+9)} = \dfrac{A}{(x+1)} + \dfrac{Bx+C}{(x^2+9)}$ ...(1)

$\Rightarrow 5x = A(x^2+9) + (Bx+C)(x+1)$

$\Rightarrow 5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$

Equating the coefficients of x^2, x, and constant term, we obtain

$A + B = 0$

$B + C = 5$

$9A + C = 0$

On solving these equations, we obtain

$A = -\dfrac{1}{2}, B = \dfrac{1}{2}, \text{ and } C = \dfrac{9}{2}$

From equation (1), we obtain

$$\frac{5x}{(x+1)(x^2+9)} = \frac{-1}{2(x+1)} + \frac{\frac{x}{2}+\frac{9}{2}}{(x^2+9)}$$

$$\int \frac{5x}{(x+1)(x^2+9)}dx = \int\left\{\frac{-1}{2(x+1)} + \frac{(x+9)}{2(x^2+9)}\right\}dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{2}\int\frac{x}{x^2+9}dx + \frac{9}{2}\int\frac{1}{x^2+9}dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\int\frac{2x}{x^2+9}dx + \frac{9}{2}\int\frac{1}{x^2+9}dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{9}{2}\cdot\frac{1}{3}\tan^{-1}\frac{x}{3}$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log(x^2+9) + \frac{3}{2}\tan^{-1}\frac{x}{3} + C$$

Question 7:

$$\frac{\sin x}{\sin(x-a)}$$

Solution:

$$\frac{\sin x}{\sin(x-a)}$$

Let $x - a = t \Rightarrow dx = dt$

$$\begin{aligned}\int \frac{\sin x}{\sin(x-a)} dx &= \int \frac{\sin(t+a)}{\sin t} dt \\ &= \int \frac{\sin t \cos a + \cos t \sin a}{\sin t} dt \\ &= \int (\cos a + \cot t \sin a) dt \\ &= t \cos a + \sin a \log|\sin t| + C_1 \\ &= (x-a)\cos a + \sin a \log|\sin(x-a)| + C_1 \\ &= x \cos a + \sin a \log|\sin(x-a)| - a\cos a + C_1 \\ &= \sin a \log|\sin(x-a)| + x \cos a + C\end{aligned}$$

Question 8:

$$\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$$

Solution:

$$\begin{aligned}\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} &= \frac{e^{4\log x}\left(e^{\log x} - 1\right)}{e^{2\log x}\left(e^{\log x} - 1\right)} \\ &= e^{2\log x} \\ &= e^{\log x^2} \\ &= x^2\end{aligned}$$

$$\therefore \int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} dx = \int x^2 dx = \frac{x^3}{3} + C$$

Question 9:

$$\frac{\cos x}{\sqrt{4 - \sin^2 x}}$$

Solution:

$$\frac{\cos x}{\sqrt{4-\sin^2 x}}$$

Let sin $x = t \Rightarrow \cos x\, dx = dt$

$$\Rightarrow \int \frac{\cos x}{\sqrt{4-\sin^2 x}} dx = \int \frac{dt}{\sqrt{(2)^2-(t)^2}}$$

$$= \sin^{-1}\left(\frac{t}{2}\right) + C$$

$$= \sin^{-1}\left(\frac{\sin x}{2}\right) + C$$

Question 10:

$$\frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x} = \frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^4 x-\cos^4 x\right)}{\sin^2 x+\cos^2 x-\sin^2 x\cos^2 x-\sin^2 x\cos^2 x}$$

$$= \frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^2 x+\cos^2 x\right)\left(\sin^2 x-\cos^2 x\right)}{\left(\sin^2 x-\sin^2 x\cos^2 x\right)+\left(\cos^2 x-\sin^2 x\cos^2 x\right)}$$

$$= \frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^2 x-\cos^2 x\right)}{\sin^2 x\left(1-\cos^2 x\right)+\cos^2 x\left(1-\sin^2 x\right)}$$

$$= \frac{-\left(\sin^4 x+\cos^4 x\right)\left(\cos^2 x-\sin^2 x\right)}{\left(\sin^4 x+\cos^4 x\right)}$$

$$= -\cos 2x$$

$$\therefore \int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x} dx = \int -\cos 2x\, dx = -\frac{\sin 2x}{2} + C$$

Solution:

$$\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}=\frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^4 x-\cos^4 x\right)}{\sin^2 x+\cos^2 x-\sin^2 x\cos^2 x-\sin^2 x\cos^2 x}$$
$$=\frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^2 x+\cos^2 x\right)\left(\sin^2 x-\cos^2 x\right)}{\left(\sin^2 x-\sin^2 x\cos^2 x\right)+\left(\cos^2 x-\sin^2 x\cos^2 x\right)}$$
$$=\frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^2 x-\cos^2 x\right)}{\sin^2 x\left(1-\cos^2 x\right)+\cos^2 x\left(1-\sin^2 x\right)}$$
$$=\frac{-\left(\sin^4 x+\cos^4 x\right)\left(\cos^2 x-\sin^2 x\right)}{\left(\sin^4 x+\cos^4 x\right)}$$
$$=-\cos 2x$$
$$\therefore \int\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}dx=\int-\cos 2x\,dx=-\frac{\sin 2x}{2}+C$$

Question 11:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Solution:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Multiplying and dividing by $\sin(a-b)$, we obtain

$$\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right]$$
$$=\frac{1}{\sin(a-b)}\left[\frac{\sin\left[(x+a)-(x+b)\right]}{\cos(x+a)\cos(x+b)}\right]$$
$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)\cdot\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right]$$
$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)}{\cos(x+a)}-\frac{\sin(x+b)}{\cos(x+b)}\right]$$
$$=\frac{1}{\sin(a-b)}\left[\tan(x+a)-\tan(x+b)\right]$$

$$\int\frac{1}{\cos(x+a)\cos(x+b)}dx=\frac{1}{\sin(a-b)}\int\left[\tan(x+a)-\tan(x+b)\right]dx$$
$$=\frac{1}{\sin(a-b)}\left[-\log\left|\cos(x+a)\right|+\log\left|\cos(x+b)\right|\right]+C$$
$$=\frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+C$$

Question 12:

$$\frac{x^3}{\sqrt{1-x^8}}$$

Solution:

$$\frac{x^3}{\sqrt{1-x^8}}$$

Let $x^4 = t \Rightarrow 4x^3\, dx = dt$

$$\Rightarrow \int \frac{x^3}{\sqrt{1-x^8}}dx = \frac{1}{4}\int \frac{dt}{\sqrt{1-t^2}}$$
$$= \frac{1}{4}\sin^{-1} t + C$$
$$= \frac{1}{4}\sin^{-1}\left(x^4\right) + C$$

Question 13:

$$\frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}$$

Solution:

$$\frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}$$

Let $e^x = t \Rightarrow e^x\, dx = dt$

$$\Rightarrow \int \frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}dx = \int \frac{dt}{(t+1)(t+2)}$$
$$= \int \left[\frac{1}{(t+1)} - \frac{1}{(t+2)}\right]dt$$
$$= \log|t+1| - \log|t+2| + C$$
$$= \log\left|\frac{t+1}{t+2}\right| + C$$
$$= \log\left|\frac{1+e^x}{2+e^x}\right| + C$$

Question 14:

$$\frac{1}{\left(x^2+1\right)\left(x^2+4\right)}$$

Solution:

$$\frac{1}{(x^2+1)(x^2+4)}$$

$$\therefore \frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+4)}$$

$$\Rightarrow 1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$

$$\Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$$

Equating the coefficients of x^3, x^2, x, and constant term, we obtain

$A + C = 0$

$B + D = 0$

$4A + C = 0$

$4B + D = 1$

On solving these equations, we obtain

$$A = 0,\ B = \frac{1}{3},\ C = 0, \text{and } D = -\frac{1}{3}$$

From equation (1), we obtain

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{1}{3(x^2+1)} - \frac{1}{3(x^2+4)}$$

$$\int \frac{1}{(x^2+1)(x^2+4)} dx = \frac{1}{3}\int \frac{1}{x^2+1} dx - \frac{1}{3}\int \frac{1}{x^2+4} dx$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{3}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C$$

Question 15:

$\cos^3 x e^{\log \sin x}$

Solution:

$\cos^3 x e^{\log \sin x} = \cos^3 x \times \sin x$

Let $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$\Rightarrow \int \cos^3 x\, e^{\log \sin x} dx = \int \cos^3 x \sin x dx$$
$$= -\int t \cdot dt$$
$$= -\frac{t^4}{4} + C$$
$$= -\frac{\cos^4 x}{4} + C$$

Question 16:

$e^{3\log x}\left(x^4+1\right)^{-1}$

Solution:

$$e^{3\log x}\left(x^4+1\right)^{-1} = e^{\log x^3}\left(x^4+1\right)^{-1} = \frac{x^3}{\left(x^4+1\right)}$$

Let $x^4+1=t \Rightarrow 4x^3\, dx = dt$

$$\Rightarrow \int e^{3\log x}\left(x^4+1\right)^{-1} dx = \int \frac{x^3}{\left(x^4+1\right)} dx$$
$$= \frac{1}{4}\int \frac{dt}{t}$$
$$= \frac{1}{4}\log|t| + C$$
$$= \frac{1}{4}\log\left|x^4+1\right| + C$$
$$= \frac{1}{4}\log\left(x^4+1\right) + C$$

Question 17:

$f'(ax+b)\left[f(ax+b)\right]^n$

Let $f(ax+b) = t \Rightarrow af'(ax+b)dx = dt$

$$\Rightarrow \int f'(ax+b)\left[f(ax+b)\right]^n dx = \frac{1}{a}\int t^n dt$$
$$= \frac{1}{a}\left[\frac{t^{n+1}}{n+1}\right]$$
$$= \frac{1}{a(n+1)}\left(f(ax+b)\right)^{n+1} + C$$

Solution:

$$f'(ax+b)\left[f(ax+b)\right]^n$$

Let $f(ax+b)=t \Rightarrow af'(ax+b)dx=dt$

$$\Rightarrow \int f'(ax+b)\left[f(ax+b)\right]^n dx=\frac{1}{a}\int t^n dt$$

$$=\frac{1}{a}\left[\frac{t^{n+1}}{n+1}\right]$$

$$=\frac{1}{a(n+1)}\left(f(ax+b)\right)^{n+1}+\text{C}$$

Question 18:

$$\frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}$$

Solution:

$$\frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}=\frac{1}{\sqrt{\sin^3 x(\sin x\cos\alpha+\cos x\sin\alpha)}}$$

$$=\frac{1}{\sqrt{\sin^4 x\cos\alpha+\sin^3 x\cos x\sin\alpha}}$$

$$=\frac{1}{\sin^2 x\sqrt{\cos\alpha+\cot x\sin\alpha}}$$

$$=\frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha+\cot x\sin\alpha}}$$

Let $\cos\alpha+\cot x\sin\alpha=t \Rightarrow -\operatorname{cosec}^2 x\sin\alpha\, dx=dt$

$$\therefore \int\frac{1}{\sin^3 x\sin(x+\alpha)}dx=\int\frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha+\cot x\sin\alpha}}dx$$

$$=\frac{-1}{\sin\alpha}\int\frac{dt}{\sqrt{t}}$$

$$=\frac{-1}{\sin\alpha}\left[2\sqrt{t}\right]+\text{C}$$

$$=\frac{-1}{\sin\alpha}\left[2\sqrt{\cos\alpha+\cot x\sin\alpha}\right]+\text{C}$$

$$=\frac{-2}{\sin\alpha}\sqrt{\cos\alpha+\frac{\cos x\sin\alpha}{\sin x}}+\text{C}$$

$$=\frac{-2}{\sin\alpha}\sqrt{\frac{\sin x\cos\alpha+\cos x\sin\alpha}{\sin x}}+\text{C}$$

$$=-\frac{2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}}+\text{C}$$

Question 20:

$$\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

Solution:

$$I = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$$

$$\text{Let } x = \cos^2\theta \Rightarrow \ dx = -2\sin\theta\cos\theta\, d\theta$$

$$I = \int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\left(-2\sin\theta\cos\theta\right)d\theta$$

$$= -\int\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin 2\theta\, d\theta$$

$$= -\int\tan\frac{\theta}{2}\cdot 2\sin\theta\cos\theta\, d\theta$$

$$= -2\int\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\theta\, d\theta$$

$$= -2\int\sin^2\theta\, d\theta + 4\int\sin^2\frac{\theta}{2}d\theta$$

$$= -2\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta + 4\int\frac{1-\cos\theta}{2}d\theta$$

$$= -2\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right] + 4\left[\frac{\theta}{2} - \frac{\sin\theta}{2}\right] + \text{C}$$

$$= -\theta + \frac{\sin 2\theta}{2} + 2\theta - 2\sin\theta + \text{C}$$

$$= \theta + \frac{\sin 2\theta}{2} - 2\sin\theta + \text{C}$$

$$= \theta + \frac{2\sin\theta\cos\theta}{2} - 2\sin\theta + \text{C}$$

$$= \theta + \sqrt{1-\cos^2\theta}\cdot\cos\theta - 2\sqrt{1-\cos^2\theta} + \text{C}$$

$$= \cos^{-1}\sqrt{x} + \sqrt{1-x}\cdot\sqrt{x} - 2\sqrt{1-x} + \text{C}$$

$$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x(1-x)} + \text{C}$$

$$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x-x^2} + \text{C}$$

Question 21:

$$\frac{2+\sin 2x}{1+\cos 2x}e^x$$

Solution:

$$I = \int \left(\frac{2 + \sin 2x}{1 + \cos 2x} \right) e^x$$
$$= \int \left(\frac{2 + 2\sin x \cos x}{2\cos^2 x} \right) e^x$$
$$= \int \left(\frac{1 + \sin x \cos x}{\cos^2 x} \right) e^x$$
$$= \int \left(\sec^2 x + \tan x \right) e^x$$

$$\text{Let } f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$$
$$\therefore I = \int \left(f(x) + f'(x) \right] e^x dx$$
$$= e^x f(x) + C$$
$$= e^x \tan x + C$$

Question 22:

$$\frac{x^2 + x + 1}{(x+1)^2 (x+2)}$$

Solution:

$$\text{Let } \frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)} \qquad \text{...(1)}$$

$$\Rightarrow x^2+x+1 = A(x+1)(x+2)+B(x+2)+C(x^2+2x+1)$$

$$\Rightarrow x^2+x+1 = A(x^2+3x+2)+B(x+2)+C(x^2+2x+1)$$

$$\Rightarrow x^2+x+1 = (A+C)x^2+(3A+B+2C)x+(2A+2B+C)$$

Equating the coefficients of x^2, x,and constant term, we obtain

$A + C = 1$

$3A + B + 2C = 1$

$2A + 2B + C = 1$

On solving these equations, we obtain

$A = -2$, $B = 1$, and $C = 3$

From equation (1), we obtain

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{-2}{(x+1)} + \frac{3}{(x+2)} + \frac{1}{(x+1)^2}$$

$$\int \frac{x^2+x+1}{(x+1)^2(x+2)}dx = -2\int \frac{1}{x+1}dx + 3\int \frac{1}{(x+2)}dx + \int \frac{1}{(x+1)^2}dx$$

$$= -2\log|x+1| + 3\log|x+2| - \frac{1}{(x+1)} + C$$

Question 23:

$$\tan^{-1}\sqrt{\frac{1-x}{1+x}}$$

Solution:

$$I = \tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$$

Let $x = \cos\theta \Rightarrow dx = -\sin\theta d\theta$

$$I = \int \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\left(-\sin\theta d\theta\right)$$

$$= -\int \tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin\theta d\theta$$

$$= -\int \tan^{-1}\tan\frac{\theta}{2}\cdot\sin\theta d\theta$$

$$= -\frac{1}{2}\int \theta\cdot\sin\theta d\theta$$

$$= -\frac{1}{2}\left[\theta\cdot\left(-\cos\theta\right) - \int 1\cdot\left(-\cos\theta\right)d\theta\right]$$

$$= -\frac{1}{2}\left[-\theta\cos\theta + \sin\theta\right]$$

$$= +\frac{1}{2}\theta\cos\theta - \frac{1}{2}\sin\theta$$

$$= \frac{1}{2}\cos^{-1}x\cdot x - \frac{1}{2}\sqrt{1-x^2} + C$$

$$= \frac{x}{2}\cos^{-1}x - \frac{1}{2}\sqrt{1-x^2} + C$$

$$= \frac{1}{2}\left(x\cos^{-1}x - \sqrt{1-x^2}\right) + C$$

Question 24:

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right) - 2\log x\right]}{x^4}$$

Solution:

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(x^2+1\right)-\log x^2\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

Let $1+\frac{1}{x^2}=t \Rightarrow \frac{-2}{x^3}dx=dt$

$$\therefore I=\int\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)dx$$

$$=-\frac{1}{2}\int\sqrt{t}\log t\,dt$$

$$=-\frac{1}{2}\int t^{\frac{1}{2}}\cdot\log t\,dt$$

Integrating by parts, we obtain

$$I=-\frac{1}{2}\left[\log t\cdot\int t^{\frac{1}{2}}dt-\int\left\{\left(\frac{d}{dt}\log t\right)\int t^{\frac{1}{2}}dt\right\}dt\right]$$

$$=-\frac{1}{2}\left[\log t\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}}-\int\frac{1}{t}\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{2}{3}\int t^{\frac{1}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{4}{9}t^{\frac{3}{2}}\right]$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\log t+\frac{2}{9}t^{\frac{3}{2}}$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\left[\log t-\frac{2}{3}\right]$$

$$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+C$$

Question 25:

$$\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

Solution:

$$I = \int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{\operatorname{cosec}^2\frac{x}{2}}{2} - \cot\frac{x}{2}\right)dx$$

$$\text{Let } f(x) = -\cot\frac{x}{2}$$

$$\Rightarrow f'(x) = -\left(-\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right) = \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$$

$$\therefore I = \int_{\frac{\pi}{2}}^{\pi} e^x\left(f(x)+f'(x)\right]dx$$

$$= \left[e^x \cdot f(x)dx\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[e^x \cdot \cot\frac{x}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[e^\pi \times \cot\frac{\pi}{2} - e^{\frac{\pi}{2}} \times \cot\frac{\pi}{4}\right]$$

$$= -\left[e^\pi \times 0 - e^{\frac{\pi}{2}} \times 1\right]$$

$$= e^{\frac{\pi}{2}}$$

Question 26:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x\cos x}{\cos^4 x+\sin^4 x}dx$$

Solution:

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\frac{(\sin x \cos x)}{\cos^4 x}}{\frac{(\cos^4 x + \sin^4 x)}{\cos^4 x}} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx$$

$$\text{Let } \tan^2 x = t \Rightarrow 2 \tan x \sec^2 x \, dx = dt$$

When $x = 0$, $t = 0$ and when $x = \frac{\pi}{4}, t = 1$

$$\therefore I = \frac{1}{2} \int_0^1 \frac{dt}{1+t^2}$$

$$= \frac{1}{2} \left[\tan^{-1} t \right]_0^1$$

$$= \frac{1}{2} \left[\tan^{-1} 1 - \tan^{-1} 0 \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{4} \right]$$

$$= \frac{\pi}{8}$$

Question 27:

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x \, dx}{\cos^2 x + 4 \sin^2 x}$$

Solution:

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4(1-\cos^2 x)} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 - 4\cos^2 x} dx$$

$$\Rightarrow I = \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 3\cos^2 x - 4}{4 - 3\cos^2 x} dx$$

$$\Rightarrow I = \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 3\cos^2 x}{4 - 3\cos^2 x} dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4}{4 - 3\cos^2 x} dx$$

$$\Rightarrow I = \frac{-1}{3} \int_0^{\frac{\pi}{2}} 1\, dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\sec^2 x - 3} dx$$

$$\Rightarrow I = \frac{-1}{3} [x]_0^{\frac{\pi}{2}} + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4(1+\tan^2 x) - 3} dx$$

$$\Rightarrow I = -\frac{\pi}{6} + \frac{2}{3} \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x} dx \qquad \ldots(1)$$

$$\text{Consider, } \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x} dx$$

$$\text{Let } 2\tan x = t \Rightarrow 2\sec^2 x\, dx = dt$$

When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = \infty$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x} dx = \int_0^{\infty} \frac{dt}{1+t^2}$$

$$= \left[\tan^{-1} t\right]_0^{\infty}$$

$$= \left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$= \frac{\pi}{2}$$

Therefore, from (1), we obtain

$$I = -\frac{\pi}{6} + \frac{2}{3}\left[\frac{\pi}{2}\right] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

Question 28:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

Solution:

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{-(-\sin 2x)}} dx$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-1+1-2\sin x\cos x)}} dx$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{1-(\sin^2 x + \cos^2 x - 2\sin x\cos x)}} dx$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)dx}{\sqrt{1-(\sin x - \cos x)^2}}$

Let $(\sin x - \cos x) = t \Rightarrow (\sin x + \cos x)dx = dt$

When $x = \frac{\pi}{6}, t = \left(\frac{1-\sqrt{3}}{2}\right)$ and when $x = \frac{\pi}{3}, t = \left(\frac{\sqrt{3}-1}{2}\right)$

$I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$

$\Rightarrow I = \int_{-\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$

As $\frac{1}{\sqrt{1-(-t)^2}} = \frac{1}{\sqrt{1-t^2}}$, therefore, $\frac{1}{\sqrt{1-t^2}}$ is an even function.

It is known that if *f*(*x*) is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

$\Rightarrow I = 2\int_0^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$

$= \left[2\sin^{-1} t\right]_0^{\frac{\sqrt{3}-1}{2}}$

$= 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$

Question 29:

$\int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}}$

Solution:

Let $I = \int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}}$

$$I = \int_0^1 \frac{1}{\left(\sqrt{1+x} - \sqrt{x}\right)} \times \frac{\left(\sqrt{1+x} + \sqrt{x}\right)}{\left(\sqrt{1+x} + \sqrt{x}\right)} dx$$

$$= \int_0^1 \frac{\sqrt{1+x} + \sqrt{x}}{1+x-x} dx$$

$$= \int_0^1 \sqrt{1+x}\, dx + \int_0^1 \sqrt{x}\, dx$$

$$= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{3}(x)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{3}\left[(2)^{\frac{3}{2}} - 1\right] + \frac{2}{3}[1]$$

$$= \frac{2}{3}(2)^{\frac{3}{2}}$$

$$= \frac{2 \cdot 2\sqrt{2}}{3}$$

$$= \frac{4\sqrt{2}}{3}$$

Question 30:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$$

Let $I = \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$

Also, let $\sin x - \cos x = t \Rightarrow (\cos x + \sin x) dx = dt$

When $x = 0$, $t = -1$ and when $x = \frac{\pi}{4}$, $t = 0$

$\Rightarrow (\sin x - \cos x)^2 = t^2$

$\Rightarrow \sin^2 x + \cos^2 x - 2\sin x \cos x = t^2$

$\Rightarrow 1 - \sin 2x = t^2$

$\Rightarrow \sin 2x = 1 - t^2$

$$\therefore I = \int_{-1}^{0} \frac{dt}{9 + 16(1 - t^2)}$$

$$= \int_{-1}^{0} \frac{dt}{9 + 16 - 16t^2}$$

$$= \int_{-1}^{0} \frac{dt}{25 - 16t^2} = \int_{-1}^{0} \frac{dt}{(5)^2 - (4t)^2}$$

$$= \frac{1}{4}\left[\frac{1}{2(5)} \log\left|\frac{5 + 4t}{5 - 4t}\right|\right]_{-1}^{0}$$

$$= \frac{1}{40}\left[\log(1) - \log\left|\frac{1}{9}\right|\right]$$

$$= \frac{1}{40} \log 9$$

Question 31:

$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx$$

Solution:

Let $I=\int_0^{\frac{\pi}{2}}\sin 2x\tan^{-1}(\sin x)\,dx=\int_0^{\frac{\pi}{2}}2\sin x\cos x\tan^{-1}(\sin x)\,dx$

Also, let $\sin x=t \Rightarrow \cos x\,dx=dt$

When $x=0, t=0$ and when $x=\frac{\pi}{2}, t=1$

$$\Rightarrow I=2\int_0^1 t\tan^{-1}(t)\,dt \qquad \ldots(1)$$

Consider $\int t\cdot\tan^{-1}t\,dt=\tan^{-1}t\cdot\int t\,dt-\int\left\{\frac{d}{dt}\left(\tan^{-1}t\right)\int t\,dt\right\}dt$

$$=\tan^{-1}t\cdot\frac{t^2}{2}-\int\frac{1}{1+t^2}\cdot\frac{t^2}{2}dt$$

$$=\frac{t^2\tan^{-1}t}{2}-\frac{1}{2}\int\frac{t^2+1-1}{1+t^2}dt$$

$$=\frac{t^2\tan^{-1}t}{2}-\frac{1}{2}\int 1\,dt+\frac{1}{2}\int\frac{1}{1+t^2}dt$$

$$=\frac{t^2\tan^{-1}t}{2}-\frac{1}{2}\cdot t+\frac{1}{2}\tan^{-1}t$$

$$\Rightarrow \int_0^1 t\cdot\tan^{-1}t\,dt=\left[\frac{t^2\cdot\tan^{-1}t}{2}-\frac{t}{2}+\frac{1}{2}\tan^{-1}t\right]_0^1$$

$$=\frac{1}{2}\left[\frac{\pi}{4}-1+\frac{\pi}{4}\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{2}-1\right]=\frac{\pi}{4}-\frac{1}{2}$$

From equation (1), we obtain

$$I=2\left[\frac{\pi}{4}-\frac{1}{2}\right]=\frac{\pi}{2}-1$$

Question 32:

$$\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}dx$$

Solution:

$$Let\ I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx \qquad \ldots(1)$$

$$I = \int_0^{\pi} \left\{ \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} \right\} dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \right)$$

$$\Rightarrow I = \int_0^{\pi} \left\{ \frac{-(\pi - x)\tan x}{-(\sec x + \tan x)} \right\} dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x} dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \tan x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x + 1 - 1}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} 1.dx - \pi \int_0^{\pi} \frac{1}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi [x]_0^{\pi} - \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi^2 - \pi \int_0^{\pi} (\sec^2 x - \tan x \sec x) dx$$

$$\Rightarrow 2I = \pi^2 - \pi [\tan x - \sec x]_0^{\pi}$$

$$\Rightarrow 2I = \pi^2 - \pi [\tan \pi - \sec \pi - \tan 0 + \sec 0]$$

$$\Rightarrow 2I = \pi^2 - \pi [0 - (-1) - 0 + 1]$$

$$\Rightarrow 2I = \pi^2 - 2\pi$$

$$\Rightarrow 2I = \pi(\pi - 2)$$

$$\Rightarrow I = \frac{\pi}{2}(\pi - 2)$$

Question 33:

$$\int_1^4 [|x-1| + |x-2| + |x-3|] dx$$

Solution:

Let $I = \int_1^4 \left[|x-1| + |x-2| + |x-3|\right] dx$

$\Rightarrow I = \int_1^4 |x-1|\,dx + \int_1^4 |x-2|\,dx + \int_1^4 |x-3|\,dx$

$I = I_1 + I_2 + I_3$ ...(1)

where, $I_1 = \int_1^4 |x-1|\,dx$, $I_2 = \int_1^4 |x-2|\,dx$, and $I_3 = \int_1^4 |x-3|\,dx$

$I_1 = \int_1^4 |x-1|\,dx$

$(x-1) \geq 0$ for $1 \leq x \leq 4$

$\therefore I_1 = \int_1^4 (x-1)\,dx$

$\Rightarrow I_1 = \left[\frac{x^2}{x} - x\right]_1^4$

$\Rightarrow I_1 = \left[8 - 4 - \frac{1}{2} + 1\right] = \frac{9}{2}$ ...(2)

$$I_2 = \int_1^4 |x-2|\,dx$$

$x-2 \geq 0$ for $2 \leq x \leq 4$ and $x-2 \leq 0$ for $1 \leq x \leq 2$

$$\therefore I_2 = \int_1^2 (2-x)\,dx + \int_2^4 (x-2)\,dx$$

$$\Rightarrow I_2 = \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

$$\Rightarrow I_2 = \left[4-2-2+\frac{1}{2}\right] + \left[8-8-2+4\right]$$

$$\Rightarrow I_2 = \frac{1}{2} + 2 = \frac{5}{2} \qquad ...(3)$$

$$I_3 = \int_1^4 |x-3|\,dx$$

$x-3 \geq 0$ for $3 \leq x \leq 4$ and $x-3 \leq 0$ for $1 \leq x \leq 3$

$$\therefore I_3 = \int_1^3 (3-x)\,dx + \int_3^4 (x-3)\,dx$$

$$\Rightarrow I_3 = \left[3x - \frac{x^2}{2}\right]_1^3 + \left[\frac{x^2}{2} - 3x\right]_3^4$$

$$\Rightarrow I_3 = \left[9 - \frac{9}{2} - 3 + \frac{1}{2}\right] + \left[8 - 12 - \frac{9}{2} + 9\right]$$

$$\Rightarrow I_3 = \left[6-4\right] + \left[\frac{1}{2}\right] = \frac{5}{2} \qquad ...(4)$$

From equations (1), (2), (3), and (4), we obtain

$$I = \frac{9}{2} + \frac{5}{2} + \frac{5}{2} = \frac{19}{2}$$

Question 34:

$$\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$$

Solution:

Let $I = \int_1^3 \frac{dx}{x^2(x+1)}$

Also, let $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$\Rightarrow 1 = Ax(x+1) + B(x+1) + C(x^2)$

$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$

Equating the coefficients of x^2, x, and constant term, we obtain

$A + C = 0$

$A + B = 0$

$B = 1$

On solving these equations, we obtain

$A = -1$, $C = 1$, and $B = 1$

$$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}$$

$$\Rightarrow I = \int_1^3 \left\{ -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx$$

$$= \left[-\log x - \frac{1}{x} + \log(x+1) \right]_1^3$$

$$= \left[\log\left(\frac{x+1}{x}\right) - \frac{1}{x} \right]_1^3$$

$$= \log\left(\frac{4}{3}\right) - \frac{1}{3} - \log\left(\frac{2}{1}\right) + 1$$

$$= \log 4 - \log 3 - \log 2 + \frac{2}{3}$$

$$= \log 2 - \log 3 + \frac{2}{3}$$

$$= \log\left(\frac{2}{3}\right) + \frac{2}{3}$$

Hence, the given result is proved.

Question 35:

$\int_0^1 xe^x dx = 1$

Solution:

Let $I = \int_0^1 xe^x \, dx$

Integrating by parts, we obtain

$$I = x\int_0^1 e^x dx - \int_0^1 \left\{\left(\frac{d}{dx}(x)\right)\int e^x dx\right\}dx$$
$$= \left[xe^x\right]_0^1 - \int_0^1 e^x dx$$
$$= \left[xe^x\right]_0^1 - \left[e^x\right]_0^1$$
$$= e - e + 1$$
$$= 1$$

Hence, the given result is proved.

Question 36:

$$\int_{-1}^1 x^{17}\cos^4 x dx = 0$$

Solution:

Let $I = \int_{-1}^1 x^{17}\cos^4 x dx$

Also, let $f(x) = x^{17}\cos^4 x$

$\Rightarrow f(-x) = (-x)^{17}\cos^4(-x) = -x^{17}\cos^4 x = -f(x)$

Therefore, $f(x)$ is an odd function.

It is known that if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

$\therefore I = \int_{-1}^1 x^{17}\cos^4 x \, dx = 0$

Hence, the given result is proved.

Question 37:

$$\int_0^{\frac{\pi}{2}} \sin^3 x \, dx = \frac{2}{3}$$

Solution:

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \sin^3 x\,dx$$

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x \cdot \sin x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \left(1-\cos^2 x\right)\sin x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x\,dx - \int_0^{\frac{\pi}{2}} \cos^2 x \cdot \sin x\,dx$$

$$= [-\cos x]_0^{\frac{\pi}{2}} + \left[\frac{\cos^3 x}{3}\right]_0^{\frac{\pi}{2}}$$

$$= 1 + \frac{1}{3}[-1] = 1 - \frac{1}{3} = \frac{2}{3}$$

Hence, the given result is proved.

Question 38:

$$\int_0^{\frac{\pi}{4}} 2\tan^3 x dx = 1 - \log 2$$

Solution:

$$\text{Let } I = \int_0^{\frac{\pi}{4}} 2\tan^3 x\,dx$$

$$I = 2\int_0^{\frac{\pi}{4}} \tan^2 x \tan x\,dx = 2\int_0^{\frac{\pi}{4}} \left(\sec^2 x - 1\right)\tan x\,dx$$

$$= 2\int_0^{\frac{\pi}{4}} \sec^2 x \tan x\,dx - 2\int_0^{\frac{\pi}{4}} \tan x\,dx$$

$$= 2\left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} + 2[\log\cos x]_0^{\frac{\pi}{4}}$$

$$= 1 + 2\left[\log\cos\frac{\pi}{4} - \log\cos 0\right]$$

$$= 1 + 2\left[\log\frac{1}{\sqrt{2}} - \log 1\right]$$

$$= 1 - \log 2 - \log 1 = 1 - \log 2$$

Hence, the given result is proved.

Question 39:

$$\int_0^1 \sin^{-1} x\,dx = \frac{\pi}{2} - 1$$

Solution:

$$\text{Let } I = \int_0^1 \sin^{-1} x\, dx$$

$$\Rightarrow I = \int_0^1 \sin^{-1} x \cdot 1 \cdot dx$$

Integrating by parts, we obtain

$$I = \left[\sin^{-1} x \cdot x\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} \cdot x\, dx$$

$$= \left[x \sin^{-1} x\right]_0^1 + \frac{1}{2}\int_0^1 \frac{(-2x)}{\sqrt{1-x^2}} dx$$

Let $1 - x^2 = t \Rightarrow -2x\, dx = dt$

When $x = 0,\ t = 1$ and when $x = 1,\ t = 0$

$$I = \left[x \sin^{-1} x\right]_0^1 + \frac{1}{2}\int_1^0 \frac{dt}{\sqrt{t}}$$

$$= \left[x \sin^{-1} x\right]_0^1 + \frac{1}{2}\left[2\sqrt{t}\right]_1^0$$

$$= \sin^{-1}(1) + \left[-\sqrt{1}\right]$$

$$= \frac{\pi}{2} - 1$$

Hence, the given result is proved.

Question 40:

Evaluate $\int_0^1 e^{2-3x} dx$ **as a limit of a sum.**

Solution:

$\text{Let } I = \int_0^1 e^{2-3x}dx$

It is known that,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+...+f(a+(n-1)h)\right]$$

$$\text{Where, } h = \frac{b-a}{n}$$

$$\text{Here, } a = 0, b = 1, \text{ and } f(x) = e^{2-3x}$$

$$\Rightarrow h = \frac{1-0}{n} = \frac{1}{n}$$

$$\therefore \int_0^1 e^{2-3x}dx = (1-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(0+h)+...+f(0+(n-1)h)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[e^2 + e^{2-3h} + ...e^{2-3(n-1)h}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{1+e^{-3h}+e^{-6h}+e^{-9h}+...e^{-3(n-1)h}\right\}\right]$$

$$= \lim_{h\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-\left(e^{-3h}\right)^n}{1-\left(e^{-3h}\right)}\right\}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[e^2 \left\{ \frac{1-e^{-\frac{3}{n}\times n}}{1-e^{-\frac{3}{n}}} \right\} \right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{e^2\left(1-e^{-3}\right)}{1-e^{-\frac{3}{n}}} \right]$$

$$= e^2\left(e^{-3}-1\right)\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{e^{-\frac{3}{n}}-1} \right]$$

$$= e^2\left(e^{-3}-1\right)\lim_{n\to\infty} \left(-\frac{1}{3}\right)\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1} \right]$$

$$= \frac{-e^2\left(e^{-3}-1\right)}{3}\lim_{n\to\infty} \left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1} \right]$$

$$= \frac{-e^2\left(e^{-3}-1\right)}{3}(1) \qquad \left[\lim_{n\to\infty} \frac{x}{e^x-1}\right]$$

$$= \frac{-e^{-1}+e^2}{3}$$

$$= \frac{1}{3}\left(e^2-\frac{1}{e}\right)$$

Question 41:

$\int \frac{dx}{e^x+e^{-x}}$ **is equal to**

Solution:

A. $\tan^{-1}\left(e^{x}\right)+C$

B. $\tan^{-1}\left(e^{-x}\right)+C$

C. $\log\left(e^{x}-e^{-x}\right)+C$

D. $\log\left(e^{x}+e^{-x}\right)+C$

Let $I=\int\frac{dx}{e^{x}+e^{-x}}dx=\int\frac{e^{x}}{e^{2x}+1}dx$

Also, let $e^{x}=t \Rightarrow e^{x}\,dx=dt$

$\therefore I=\int\frac{dt}{1+t^{2}}$

$=\tan^{-1}t+C$

$=\tan^{-1}\left(e^{x}\right)+C$

Hence, the correct answer is A.

Question 42:

$\int\frac{\cos 2x}{(\sin x+\cos x)^{2}}dx$ **is equal to**

Solution:

A. $\dfrac{-1}{\sin x+\cos x}+C$

B. $\log|\sin x+\cos x|+C$

C. $\log|\sin x-\cos x|+C$

D. $\dfrac{1}{(\sin x+\cos x)^2}$

Let $I=\dfrac{\cos 2x}{(\cos x+\sin x)^2}$

$$I=\int\frac{\cos^2 x-\sin^2 x}{(\cos x+\sin x)^2}dx$$

$$=\int\frac{(\cos x+\sin x)(\cos x-\sin x)}{(\cos x+\sin x)^2}dx$$

$$=\int\frac{\cos x-\sin x}{\cos+\sin x}dx$$

Let $\cos x+\sin x=t \Rightarrow (\cos x-\sin x)dx=dt$

$$\therefore I=\int\frac{dt}{t}$$

$$=\log|t|+C$$

$$=\log|\cos x+\sin x|+C$$

Hence, the correct answer is B.

Question 43:

If $f(a+b-x)=f(x)$**, then** $\int_a^b x\,f(x)\,dx$ **is equal to**

Solution:

A. $\frac{a+b}{2}\int_a^b f(b-x)dx$

B. $\frac{a+b}{2}\int_a^b f(b+x)dx$

C. $\frac{b-a}{2}\int_a^b f(x)dx$

D. $\frac{a+b}{2}\int_a^b f(x)dx$

Let $I=\int_a^b x\,f(x)dx \qquad ...(1)$

$$I=\int_a^b (a+b-x)f(a+b-x)dx \qquad \left(\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right)$$

$$\Rightarrow I=\int_a^b (a+b-x)f(x)dx$$

$$\Rightarrow I=(a+b)\int_a^b f(x)dx \quad -I \qquad [\text{Using}(1)]$$

$$\Rightarrow I+I=(a+b)\int_a^b f(x)dx$$

$$\Rightarrow 2I=(a+b)\int_a^b f(x)dx$$

$$\Rightarrow I=\left(\frac{a+b}{2}\right)\int_a^b f(x)dx$$

Hence, the correct answer is D.

Question 44:

The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ **is**

Solution:

A. 1

B. 0

C. - 1

D. $\frac{\pi}{4}$

Let $I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$

$\Rightarrow I = \int_0^1 \tan^{-1}\left(\frac{x-(1-x)}{1+x(1-x)}\right)dx$

$\Rightarrow I = \int_0^1 \left[\tan^{-1}x - \tan^{-1}(1-x)\right]dx \qquad ...(1)$

$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(1-1+x)\right]dx$

$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(x)\right]dx$

$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(x)\right]dx \qquad ...(2)$

Adding (1) and (2), we obtain

$2I = \int_0^1 \left(\tan^{-1}x + \tan^{-1}(1-x) - \tan^{-1}(1-x) - \tan^{-1}x\right)dx$

$\Rightarrow 2I = 0$

$\Rightarrow I = 0$

Hence, the correct answer is B.